

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

- 1) ☒ T ☐ F The quadratic surface $x^2 + y - z^2 = -5$ is a hyperbolic paraboloid.

Solution:

Write it as $y - 5 = z^2 - x^2$, to see it better.

- 2) ☐ T ☒ F There are vectors $\vec{u}$ and $\vec{v}$ such that $|\vec{u} \times \vec{v}| > |\vec{u}||\vec{v}|$.

Solution:

We have a general identity $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\alpha)$ which contradicts the claim.

- 3) ☐ T ☒ F $\int_0^{2\pi} \int_0^5 r \, d\theta \, dr$ is the area of a disc of radius 5.

Solution:

There seems nothing wrong with that. But note that the 0 to 5 integration is paired with the $d\theta$.

- 4) ☒ T ☐ F If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = Q_x - P_y = 0$ for all points (x, y) in the plane, then $\vec{F}$ is a gradient field.

Solution:

True. We have derived this from Green's theorem.

- 5) ☒ T ☐ F The jerk of a parameterized curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is parallel to the acceleration if the curve $\vec{r}(t)$ is a line.

Solution:

The velocity, the acceleration and the jerk are all parallel on a line.

- 6) ☒ T ☐ F The curvature of the curve $\vec{r}(t) = \langle 3 \sin(t), 0, 3 \cos(t) \rangle$ is twice the curvature of the curve $\vec{s}(t) = \langle 6 + 6 \sin(t), 6 \cos(t), 0 \rangle$.

Solution:

If we scale a curve by a factor 2, its curvature is divided by 2.

- 7) ☒ T ☐ F The curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.

Solution:

Indeed, one can check that $x(t)^2 + z(t)^2 = 1$.

- 8) ☐ T ☒ F If a function $f(x, y)$ has the property that $f_x(x, y)$ is zero for all x, y , then f is the constant function.

Solution:

No, for example $f(x, y) = y$ is also a solution too and this solution is not constant.

- 9) ☒ T ☐ F If the unit tangent vector $\vec{T}(t)$ of a curve $\vec{r}(t)$ is always parallel to a plane Σ , then the curve is contained in a plane parallel to Σ .

Solution:

Indeed, we never leave the plane which goes through the initial point $r(0)$ because also $r'(t)$ is always parallel to Σ and after integration, the curve $r(t)$ has to be in the plane.

- 10) ☒ T ☐ F If (x_0, y_0) is an extremum of $f(x, y)$ under the constraint $x^2 + y^2 = 1$, then the same point is an extremum of $10f(x, y)$ under the same constraint.

Solution:

The point is a solution to the same Lagrange equations.

- 11) ☐ T ☒ F At a critical point (x_0, y_0) of a function $f(x, y)$ for which $f_{xx}(x_0, y_0) > 0$, the critical point is always a minimum.

Solution:

No, we also need $D > 0$.

- 12) ☒ T ☐ F If a vector field $\vec{F}(x, y)$ is a gradient field, and C is a closed curve which looks like a figure 8, then $\int_C \vec{F} \cdot d\vec{r}$ is zero.

Solution:

This follows from the fundamental theorem of line integrals.

- 13)

T	F
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 If C is part of a level curve of a function $f(x, y)$ and $\vec{F} = \langle f_x, f_y \rangle$ is the gradient field of f , then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:

The gradient field is perpendicular to the level curves.

- 14)

T	F
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 The divergence of the gradient vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is always the zero function.

Solution:

The divergence of the gradient of f is the Laplacian of f

- 15)

T	F
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 The line integral of the vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a line segment from $(0, 0, 0)$ to $(1, 1, 1)$ is $3/2$.

Solution:

By the fundamental theorem of line integrals, we can take the difference of the potential $f(x, y, z) = x^2/2 + y^2/2 + z^2/2$, which is $1/2 + 1/2 + 1/2 = 3/2$.

- 16)

T	F
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 The area of a region G can be expressed as a line integral along its boundary.

Solution:

This is a consequence of Green's theorem and we have seen a few examples.

- 17)

T	F
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 The flux of the vector field $\vec{F}(x, y, z) = \langle x, y, -z \rangle$ through the boundary S of a solid ellipsoid E is equal to the volume the ellipsoid.

Solution:

Indeed the divergence of the field is 1 and we can apply the divergence theorem.

- 18)

T	F
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 If $\vec{F}$ is a vector field in space and S is a torus surface, then the flux of $\text{curl}(\vec{F})$ through S is 0.

Solution:

This is true by Stokes theorem.

- 19) T F If the divergence and the curl of a vector field $\vec{F}$ are both zero, then it is a constant field.

Solution:

Take $\vec{F}(x, y, z) = \langle x, -y, 0 \rangle$. It has zero curl and zero divergence but is not constant.

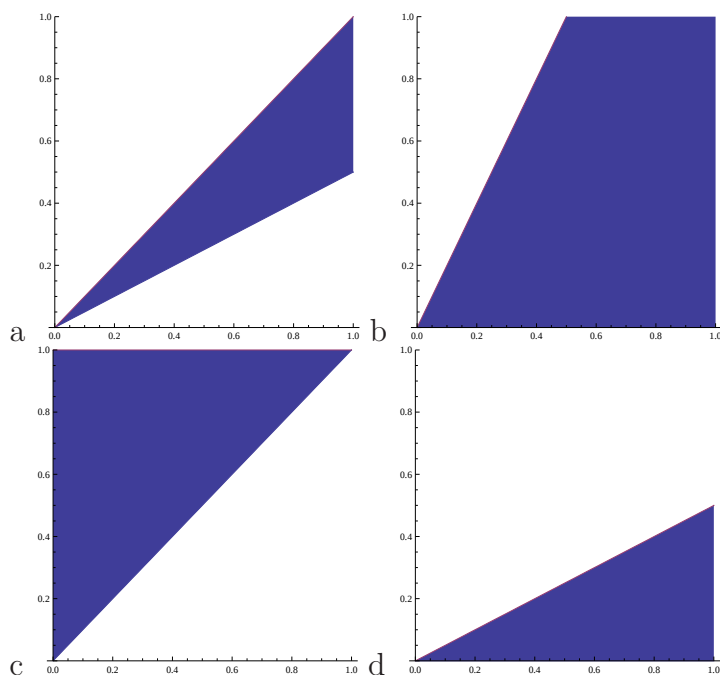
- 20) T F For any function f , the curl of $\vec{F} = \text{grad}(f)$ is the zero field $\langle 0, 0, 0 \rangle$.

Solution:

$\text{curl}(\text{grad}(f)) = \langle 0, 0, 0 \rangle$ is an important identity.

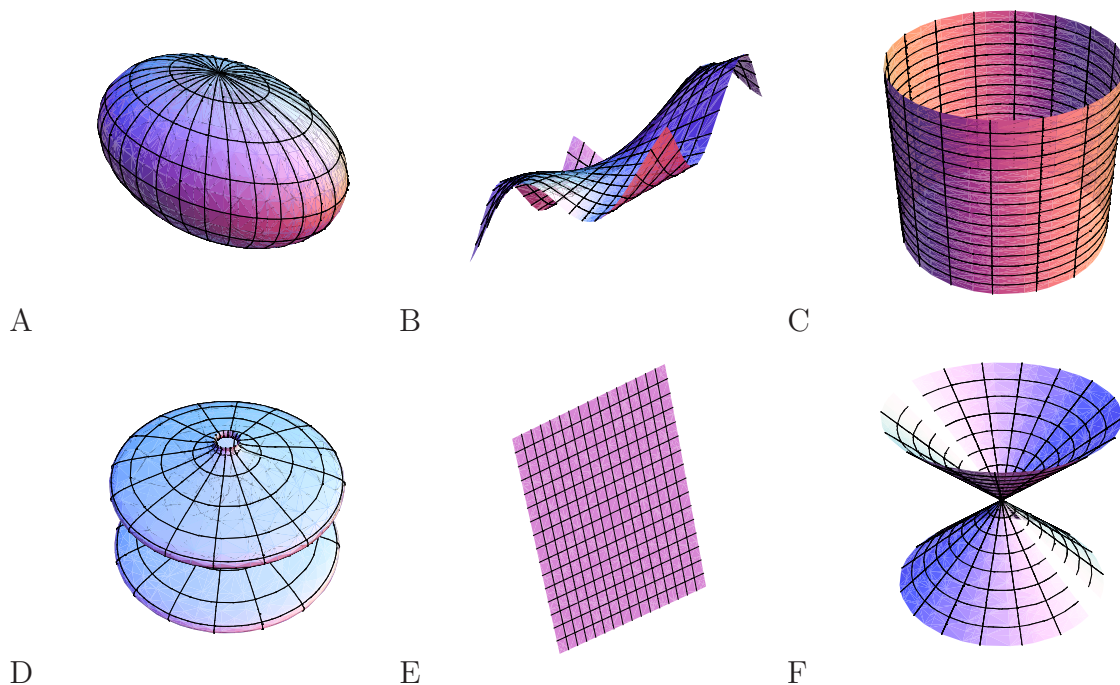
Problem 2) (10 points)

a) (4 points) Match the regions with the corresponding double integrals



Enter a,b,c,d	Function
	$\int_0^1 \int_{x/2}^x f(x, y) \, dy dx$
	$\int_0^1 \int_0^y f(x, y) \, dx dy$
	$\int_0^1 \int_0^{x/2} f(x, y) \, dy dx$
	$\int_0^1 \int_{y/2}^1 f(x, y) \, dx dy$

b) (6 points) Match the parametrized or implicit surfaces with their definitions



Enter A-F here	Function or parametrization
	$\vec{r}(u, v) = \langle \cos(u), \sin(u), v \rangle$
	$\vec{r}(u, v) = \langle u - v, u + 2v, 2u + 3v \rangle$
	$x^2 + y^2/3 + z^2/3 = 1$
	$\vec{r}(u, v) = \langle (\sin(v) + 1) \cos(u), (\sin(v) + 1) \sin(u), v \rangle$
	$z - x + \sin(xy) = 0$
	$x^2 + y^2 - z^2 = 0$

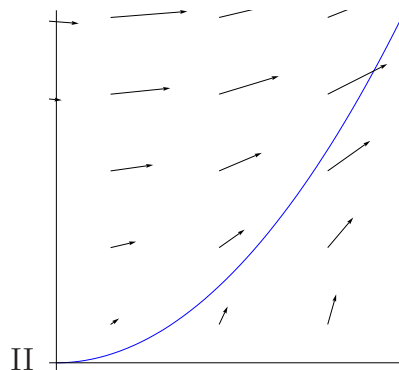
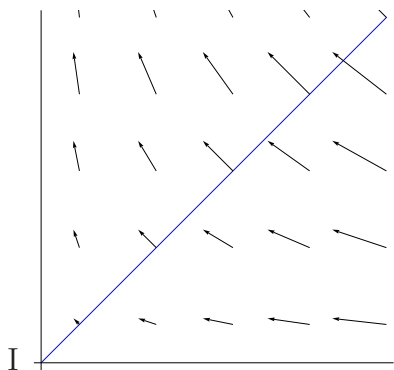
Solution:

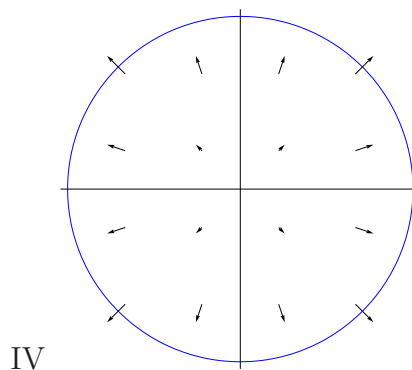
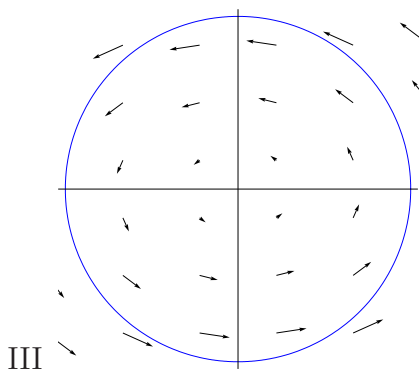
a) a c d b

b) C E A D B F

Problem 3) (10 points)

a) (4 points) Match the vector fields and curves with the corresponding line integral





Enter I,II,III,IV	Line integral
	$\int_0^{2\pi} \langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$
	$\int_0^{2\pi} \langle -t, t^2 \rangle \cdot \langle 1, 1 \rangle dt$
	$\int_0^{2\pi} \langle t^2, t \rangle \cdot \langle 1, 2t \rangle dt$
	$\int_0^{2\pi} \langle -3 \sin(t), 3 \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$

b) (6 points) Fill in from following choice: "arc length", "surface area", "chain rule", "volume of parallelepiped", "area of parallelogram", "line integral", "flux integral", "curvature".

Formula	Name of formula or rule or theorem
$\int \int_R \vec{r}_u \times \vec{r}_v du dv$	
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$	
$\int_a^b \vec{r}'(t) dt$	
$\frac{ \vec{r}'(t) \times \vec{r}''(t) }{ \vec{r}'(t) ^3}$	
$ \vec{u} \cdot (\vec{v} \times \vec{w}) $	
$\int_0^1 \int_0^1 \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$	

Solution:

a) IV, I, II, III

b) surface area, chain rule, arc length, curvature, volume of parallelepiped, flux integral.

Problem 4) (10 points)

Given the line $x - 1 = y - 2 = z - 3$ and the point $P = (8, 4, 5)$. Find the equation

$$ax + by + cz = d$$

of the plane which contains the line and the point.

Solution:

From the symmetric equation, we see that the line contains the vector $\langle 1, 1, 1 \rangle$. The line also contains the point $(1, 2, 3)$ so that the vector $\langle 7, 2, 2 \rangle$ is also in the plane. The cross product $\vec{n} = \langle 1, 1, 1 \rangle \times \langle 7, 2, 2 \rangle$ is $\langle 0, 5, -5 \rangle$ so that the equation of the plane is $5y - 5z = d$. The constant d can be obtained by plugging in a point in the plane, for example $(1, 2, 3)$ so that $d = -5$. The equation of the plane is $5y - 5z = -5$ which can also be written as $z - y = 1$.

Problem 5) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

Solution:

The critical points are $P = (-2, 0)$ and $Q = (-2, 2)$. The discriminant D at P is $D = -12$ so that P is a saddle point. The Hessian at Q is 12 and $f_{xx} = 2$ which is a local minimum.

$P = (-2, 0)$	$D = -12$	irrelevant	saddle point
$Q = (-2, 2)$	$D = 12$	$f_{xx} = 2$	local minimum

Problem 6) (10 points)

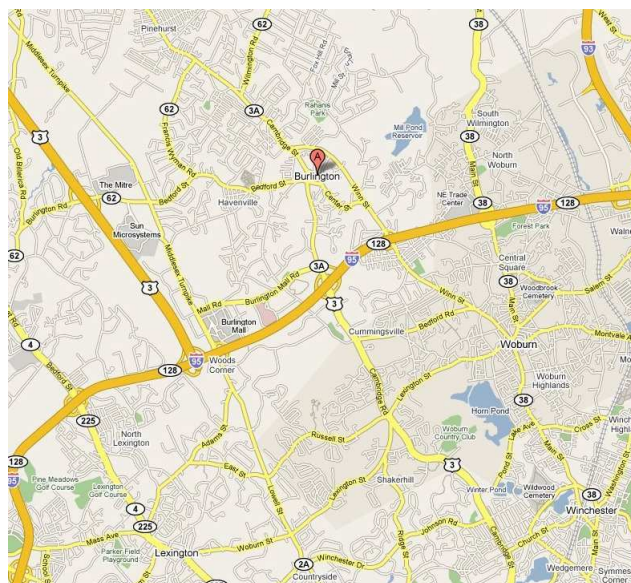
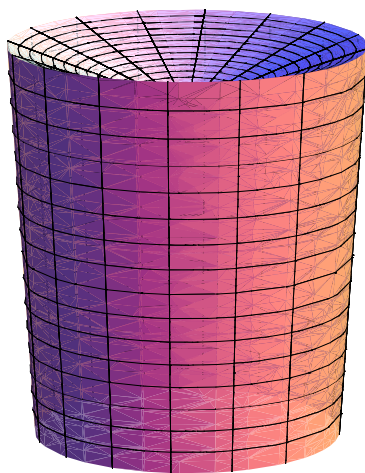
The hyperbolic paraboloid $x^2 - y^2 - 3z = 0$ contains the point $P = (1, 1, 0)$ and the point $Q = (3, 0, 3)$. Find the tangent planes to the surface at P and Q and find a parametrization $\vec{r}(t)$ of the line of intersection of these two planes.

Solution:

The gradient of $g(x, y, z) = x^2 - y^2 - z = 0$ is $(2x, -2y, -3)$ which is $(2, -2, -3)$ at the first point and $(6, 0, -3)$ at the second point. The first tangent plane equation is $2x - 2y - 3z = 0$. The second tangent plane equation is $6x - 3z = 9$. To get the line of intersection, we find a point in the intersection, like $(0, 9/2, -3)$ and the cross product $\langle 2, -2, -3 \rangle \times \langle 6, 0, -3 \rangle = \langle 6, -12, 12 \rangle$ of the normal vectors to get a direction in the line. $\vec{r}(t) = (0, 9/2, 3) + t\langle 1, -1, 2 \rangle$. Note that there are many solutions since the initial position on the line and the length of the vector in the line can change.

Problem 7) (10 points)

A water reservoir in Burlington, MA (the map to the right is centered there) is bounded by a solid cylinder $x^2 + y^2 \leq 1$. It has as the roof the cone $x^2 + y^2 = (z - 6)^2$ and is bounded from below by the xy -plane $z = 0$. What is the volume of the reservoir?

**Solution:**

If R denotes the unit disc, we have the triple integral

$$\int \int_R \int_0^{6+\sqrt{x^2+y^2}} 1 \, dz \, dy \, dx$$

which of course is integrated best in cylindrical coordinates:

$$\int_0^1 \int_0^{2\pi} \int_0^{6-r} r \, dz \, d\theta \, dr$$

which is $2\pi \int_0^1 (6r - r^2) \, dr = 2\pi(3 - 1/3) = \boxed{20\pi/3}$.

Problem 8) (10 points)

Find the maxima and minima of the function $f(x, y) = x^2 - y^2$ on the parabola $x + y^2 = 1$ using the Lagrange multiplier method.

Solution:

The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \\ -2y &= 2\lambda y \\ x + y^2 &= 1 \end{aligned}$$

The second equation can be solved by $y = 0$, a case which was often forgotten. In that case the third equation gives $x = 1$.

If y is not zero, we can divide the second equation by y . Dividing now the second by the first equation gives $-1/x = 2$ so that $x = -1/2$. Plugging into the third gives $y = \pm\sqrt{3/2}$. The solutions are $(1, 0)$, $(-1/2, -\sqrt{3/2})$, $(-1/2, \sqrt{3/2})$. By plugging in f values, we see that the first is a maximum, the other two points are minima. As usual with Lagrange methods we can not use the second derivative test to check for maxima and minima. In summary:

$(1, 0)$ is a maximum, and $(-1/2, \sqrt{3/2})$ as well as $(-1/2, -\sqrt{3/2})$ are minima.

Problem 9) (10 points)

Compute the surface area of the surface $\vec{r}(u, v) = \langle u^3, v^3, u^3 - v^3 \rangle$ parametrized so that (u, v) is in the unit disc.

Solution:

$\vec{r}_u \times \vec{r}_v = \langle -9u^2v^2, 9u^2v^2, 9u^2v^2 \rangle$ with length $|\vec{r}_u \times \vec{r}_v| = \sqrt{3}9u^2v^2$. Integrating this over the unit disc R leads to a double integral problem

$$9\sqrt{3} \int \int_R u^2 v^2 \, du dv .$$

which is best solved in polar coordinates $u = r \cos(\theta), v = r \sin(\theta)$. Using $2^2 \sin^2(\theta)^2 \cos^2(\theta)^2 = \sin^2(2\theta)^2$ we get

$$9\sqrt{3} \int_0^{2\pi} \int_0^1 r^5 (1/4) \sin^2(2\theta) \, d\theta dr$$

Using $\int_0^{2\pi} \sin^2(2\theta) \, d\theta = \pi$ we end up with $\sqrt{3}3\pi/8$.

Problem 10) (10 points)

Evaluate the following double integral

$$\int_0^2 \int_{x/2}^1 \cos(y^2) \, dy \, dx .$$

Solution:

The integral can not be evaluated as given. Change the order of integration

$$\int_0^1 \int_0^{2y} \cos(y^2) \, dx dy$$

This can now be solved. After evaluating the trivial inner integral we get

$$\int_0^1 2y \cos(y^2) \, dy = \sin(y^2)_0^1 = \sin(1) .$$

The final answer is $\sin(1)$.

Problem 11) (10 points)

Find the value of the line integral

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt ,$$

where $\vec{F}(x, y) = \langle y + \sin(\cos(x)), -2x \rangle$ and C is the boundary of the unit circle traversed in the counter clockwise direction.

Solution:

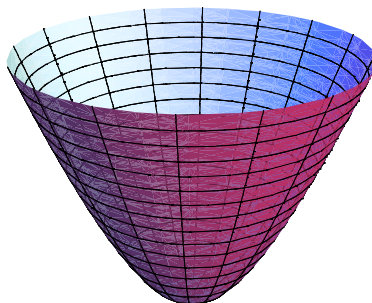
By Greens theorem the line integral is the double integral of $\text{curl}(\vec{F} = Q_x - P_y = -3$ over the unit disc, which is -3 times the area π of the disc. The final result is $\boxed{-3\pi}$.

Problem 12) (10 points)

Find the value of the flux integral

$$\int \int_S \text{curl}(\vec{F})(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \, du dv$$

where $\vec{F}(x, y, z) = \langle -y, x, z \rangle$ and S is the part of the two-sheeted hyperboloid $x^2 + y^2 - z^2 = -1$ which satisfies $1 < z < 2$ and which is oriented so that the normal vector points downwards on S .



Solution:

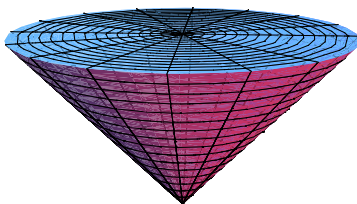
By Stokes theorem, we can compute the line integral of $\vec{F}$ along the boundary of the surface S instead. This boundary C is parametrized by $\vec{r}(t) = \langle \sqrt{3} \cos(t), \sqrt{3} \sin(t), 2 \rangle$. The line integral $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$ which is $\boxed{6\pi}$.

Problem 13) (10 points)

Let E be the solid which is bounded on the side by the cone $S_1 : x^2 + y^2 = z^2, 0 < z < 1$ and on top by the disc $S_2 = x^2 + y^2 \leq 1, z = 1$. Let $\vec{F}(x, y, z) = \langle 1 + 4x, 2 - 5y, 3 + 2z \rangle$. Find the value of the flux integral

$$\int_S \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv ,$$

where S is the union of the two surfaces S_1 and S_2 . The normal vector of S is oriented outwards on $S_1 \cup S_2$.

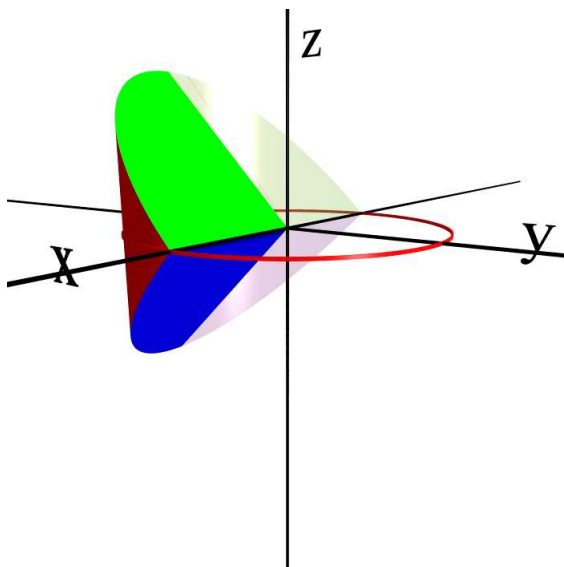


Solution:

By the divergence theorem, the flux is $\iiint_E \text{div} \vec{F} \, dxdydz$, where E is the cone. The divergence of $\vec{F}$ is constant 1 so that we have to integrate 1 over E which is the volume of the cone. This is best computed in cylindrical coordinates: $\int_0^{2\pi} \int_0^1 \int_0^z r \, drdzd\theta =$ which is $\boxed{\pi/3}$.

Problem 14) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Solution:

We use cylindrical coordinates. The base region in the xy plane is the fourth quadrant. Its roof is $z = -y$, its floor is $z = y$. We have to integrate $f(x, y) = -2y = -2r \sin(\theta)$ over the fourth quadrant and get:

$$\int_0^1 \int_{3\pi/2}^{2\pi} -2r \sin(\theta) r \, d\theta dr = 2/3 .$$

The volume of the cheese is $\boxed{2/3}$.