

Abstract

Welcome to the Harvard summer school! The material of this multivariable course is arranged in 6 chapters and delivered in 6 weeks. Each chapter has 4 sections. We cover two sections each day. While you can also consult with text books or online resources if needed, make sure to focus on lectures, homework and practice exams. Working on problems is important. Mathematics is learned best by doing it. Here are one paragraph summaries:

Chapter 1. Geometry and Space

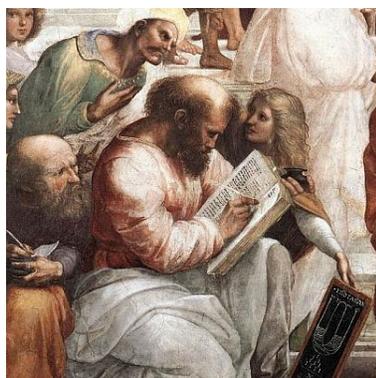
Section 1.1: Space and Distance



René Descartes

After introducing ourselves, reviewing the syllabus and administrative things, we use **coordinates** like $P = (3, 4, 5)$ to describe points P in space. As promoted by **René Descartes** in the 16'th century, geometry can be described algebraically by using a **coordinate system**. The **distance** between two points $P(x, y, z)$ and $Q = (a, b, c)$ is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$. This definition is motivated by **Pythagoras theorem**. We will prove the later. To become more familiar with space, we look at geometric objects in the plane and in space. We will focus on **cylinders** and **spheres** and learn how to find the **center** and **radius** of a sphere given as a quadratic expression in x, y, z . This is the **completion of the square**.

Section 1.2: Vectors and Dot product



Pythagoras

Two points P, Q define a **vector** \vec{PQ} . Vectors can describe **velocities**, **forces** or **color** or **data**. The **components** of \vec{PQ} connecting a point $P = (a, b, c)$ with $Q = (x, y, z)$ are the entries of the vector $\langle x - a, y - b, z - c \rangle$. Examples are the **zero vector** $\vec{0} = \langle 0, 0, 0 \rangle$, and the **standard basis vectors** $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle$. **Addition**, **subtraction** and **scalar multiplication** work both geometrically and algebraically. The **dot product** $\vec{v} \cdot \vec{w}$ is a **scalar** which gives the **length** $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. The direction $\vec{v}/|\vec{v}|$ is defined if $|\vec{v}| \neq 0$. The **angle** is defined by $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \alpha$ and justified by Cauchy-Schwarz $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$. If $\vec{v} \cdot \vec{w} = 0$, then \vec{v}, \vec{w} are **perpendicular**. **Pythagoras** $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ for perpendicular vectors and Al-Kashi's cos-formula follow.

Section 1.3: Cross and Triple Product



Rowan Hamilton

The **cross product** $\vec{v} \times \vec{w}$ of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\langle br - cq, cp - ar, aq - bp \rangle$. It is perpendicular to $\langle a, b, c \rangle$ and $\langle p, q, r \rangle$. In two dimensions, the cross product is a scalar $\langle a, b \rangle \times \langle p, q \rangle = aq - bp$. The product is useful to compute **areas** of parallelograms, the **distance** between a point and a line, or to **construct** a plane through three points or to **intersect** two planes. We prove a formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ which allows us to define the **area** of the parallelepiped spanned by \vec{v} and \vec{w} . The **triple scalar product** $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is a scalar and defines the **signed volume** of the parallelepiped spanned by \vec{u}, \vec{v} and \vec{w} . Its sign informs about the **orientation** of the coordinate system defined by the three vectors. The triple scalar product is zero if and only if the three vectors are in a common plane.

Section 1.4: Lines and Planes



Arthur Cayley

Because $\langle a, b, c \rangle = \vec{n} = \vec{u} \times \vec{v}$ is perpendicular to $\vec{x} - \vec{w}$, if \vec{x}, \vec{w} are in the plane spanned by \vec{u} and \vec{v} , points on a plane satisfy $ax + by + cz = d$. We often know the normal vector $\vec{n} = \langle a, b, c \rangle$ to a plane and can determine the constant d by plugging in a known point (x, y, z) on equation $ax + by + cz = d$. The parametrization $\vec{x}(t, s) = \vec{w} + t\vec{u} + s\vec{v}$ is an other way to represent surfaces. We introduce **lines** by the parameterization $\vec{r}(t) = \vec{O}P + t\vec{v}$, where P is a point on the line and $\vec{v} = \langle a, b, c \rangle$ is a vector telling the direction of the line. If $P = (o, p, q)$, then $(x - o)/a = (y - p)/b = (z - q)/c$ is called the **symmetric equation** of a line. It can be interpreted as the intersection of two planes. As an application of the dot and cross products, we look at various **distance formulas**.

Chapter 2. Curves and Surfaces

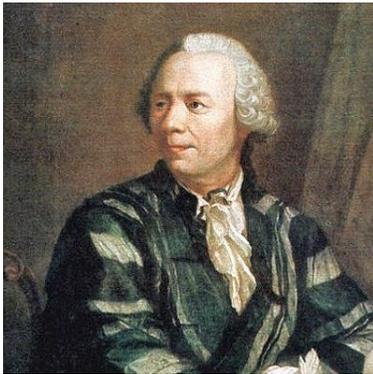
Section 2.1: Level Curves and Surfaces



Claudius Ptolemy

The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $g(x, y, z) = z - f(x, y) = 0$. We look at examples and match some graphs with functions $f(x, y)$. **Generalized traces** like $f(x, y) = c$ are called **level curves** of f and help to visualize surfaces. The set of all level curves forms a **contour map**. After a short review of **conic sections** like **ellipses**, **parabola** and **hyperbola** in two dimensions, we look at more general surfaces of the form $g(x, y, z) = 0$. We start with the **sphere** and the **plane**. If $g(x, y, z)$ is a function which only involves linear and quadratic terms, the level surface is called a **quadric**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **paraboloids**, **cylinders** as well as **hyperboloids**.

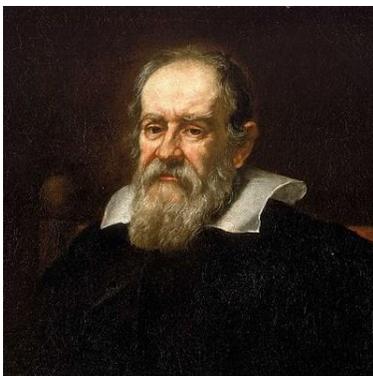
Section 2.2: Parametric Surfaces



Leonhard Euler

Surfaces are described implicitly or parametrically. Examples of implicit descriptions $g(x, y, z) = 0$ are $x^2 + y^2 + z^2 - 1 = 0$. Examples of **parametrizations** $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ are the **sphere** $\vec{r}(\theta, \phi) = \langle \rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi) \rangle$ where ρ is fixed and ϕ, θ are the **Euler angles**. Using computers, one can **visualize** also complicated surfaces. Parametrization of surfaces is important in **geodesy**, where they appear as maps or in **computer generated imaging**, where the parameterization $\vec{r}(u, v)$ is called the "**uv-map**". Parametrizations of surfaces make use of **cylindrical coordinates** (r, θ, z) , where $r \geq 0$ is the distance to the z -axes and $0 \leq \theta < 2\pi$ is an angle. **spherical coordinates** (ρ, θ, ϕ) use ρ , distance to the origin and θ, ϕ , the Euler angles.

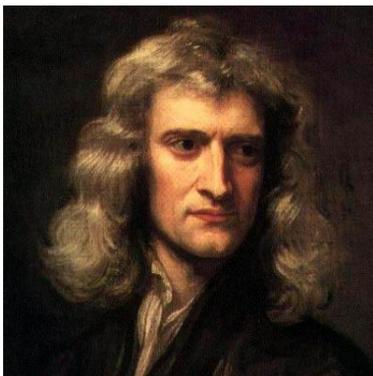
Section 2.3: Parametric Curves



Johannes Kepler

We define curves by parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, where the **parameter** t is in some parameter interval $I = [a, b]$. A parametrization has more information than the curve itself. It is a dynamical and constructive description which also tells, how the curve is traced if t is interpreted a **time variable**. Other examples of curves are **grid curves** on parametrized surfaces obtained by fixing one of the two uv parameters. Differentiation of a parametrization $\vec{r}(t)$ leads to the **velocity** $\vec{r}'(t)$, a vector which is tangent to the curve at $\vec{r}(t)$. A second differentiation with respect to t gives the **acceleration** vector $\vec{r}''(t)$. The **speed** $|\vec{r}'(t)|$ is a scalar. We also learn how to get from $\vec{r}''(t)$ and $\vec{r}'(0)$ and $\vec{r}(0)$ the position $\vec{r}(t)$ by integration. A special case is the **free fall**, where the acceleration vector is constant.

Section 2.4: Arc length and Curvature



Isaac Newton

The **arc length** of a curve is defined as a limiting length of polygons and leads to the **arc length** integral $\int_a^b |\vec{r}'(t)| dt$. A re-parametrization of a curve does not change the arc length. The **curvature** $\kappa(t)$ of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity which does not depend on parameterizations. One "feels" the acceleration and "sees" the curvature $\kappa(t) = |\vec{T}''(t)|/|\vec{T}'(t)| = |\vec{r}'(t) \times \vec{r}''(t)|/|\vec{r}'(t)|^3$ where $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$ is the **unit tangent vector** \vec{T} . Together with **normal vector** \vec{N} and **bi-normal vector** \vec{B} these three vectors form an orthonormal frame.

Chapter 3. Linearization and Gradient

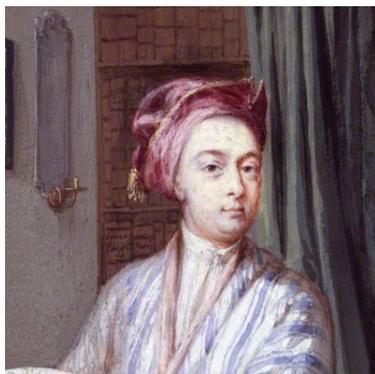
Section 3.1: Partial Derivatives



Alexis Clairot

In higher dimensions continuity is more interesting than in single variable. We look at examples of such **catastrophes** and assume in general all functions to be **smooth**: we can take as many derivatives in an variables as we wish. We introduce then **partial derivatives** $f_x = \partial_x f = \frac{\partial f}{\partial x}$ and derive **Clairot's theorem** $f_{xy} = f_{yx}$ for smooth functions. This allows to compute expressions like f_{xxyx} for $f(x, y) = x^2 \sin(\sin(9y)) + x \tan(y)$ very fast. To practice differentiation and see the use of calculus in science, we look at some **partial differential equations**, (**PDE's**). Examples are the **transport equation** $f_x(t, x) = f_t(t, x)$, the **wave equation** $f_{tt}(t, x) = f_{xx}(t, x)$ and the **heat equation** $f_t(t, x) = f_{xx}(t, x)$.

Section 3.2 Linear Approximation



Brook Taylor

Linearization is an important concept in science because many physical laws are linearization of more complicated laws. Linearization is also useful to estimate quantities. After a review of linearization of functions of one variables, we introduce the **linearization** of a function $f(x, y)$ of two variables at a point (p, q) . It is defined as the function $L(x, y) = f(p, q) + f_x(p, q)(x - p) + f_y(p, q)(y - q)$. The **tangent line** $ax + by = d$ at a point (p, q) is a level curve of L and $a = f_x, b = f_y$. Linearization works similarly in three dimensions, where it allows to compute the **tangent plane** $ax + by + cz = d$. The key is the **gradient** $\nabla f = \langle f_x, f_y, f_z \rangle$.

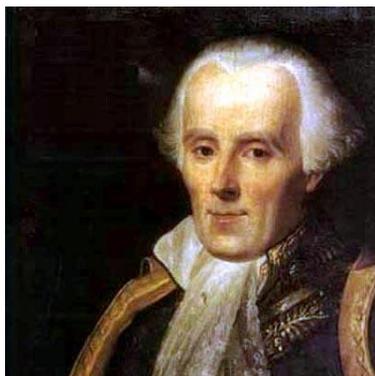
Section 3.3: Implicit Differentiation



Gottfried Wilhelm Leibniz

The **chain rule** $d/dt f(g(t)) = f'(g(t))g'(t)$ in one dimension is generalized to higher dimensions. It becomes $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$, where $\nabla f = \langle f_x, f_y, f_z \rangle$ is the gradient. Written out, this formula is $d/dt f(x(t), y(t), z(t)) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t)$. All other chain rule versions can be derived from this. A nice application of the chain rule is **implicit differentiation**: if $f(x, y, z) = 0$ defines a surface which looks locally like $z = g(x, y)$ and because $f_x + f_z z' = 0$ we can compute the partial derivatives $g_x = -f_x/f_z$ and $g_y = -f_y/f_z$ of g without knowing g .

Section 3.4: Steepest Ascent



Pierre-Simon Laplace

The **gradient** of a function is useful for the geometry of a surface $g(x, y, z) = 0$ because it is perpendicular to the level surface $f(x, y, z) = c$ at that point. One can see this both using linearization or by using the chain rule for a curve $\vec{r}(t)$ on the surface $f(\vec{r}(t)) = 0$. A special case is the plane $g(x, y, z) = ax + by + cz = d$, where $\nabla g = \langle a, b, c \rangle$. The gradient helps to find tangent planes and tangent lines. We introduce the **directional derivative** $D_{\vec{v}}(f)$ as $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ for unit vectors \vec{v} . Partial derivatives are special directional derivatives. The direction of the normal vector gives a nonnegative partial derivative. Moving into the direction of the normal vector, increases f because $D_{\nabla f/|\nabla f|}f = |\nabla f|$. In other words, the gradient vector points in the direction of **steepest ascent**.

Chapter 4. Extrema and Double integrals

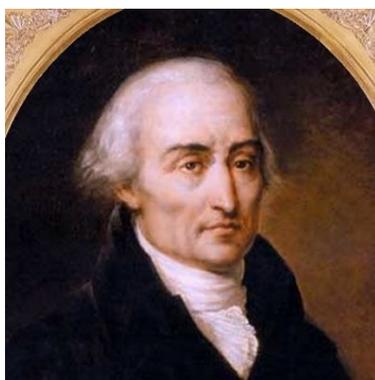
Section 4.1: Maxima and Minima



Pierre de Fermat

To **extremize** functions $f(x, y)$ of two variables, we first identify **critical points**, where the gradient $\nabla f(x, y)$ vanishes. The nature of these critical points (p, q) can often be established using the **second derivative test**. Let (p, q) be a critical point and let $D = f_{xx}f_{yy} - f_{xy}^2$ denote the **discriminant** of f at this critical point. There are three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**. If $D < 0$, then (p, q) is a saddle point, if $D > 0$ and $f_{xx} < 0$ then we have a local maximum, if $D > 0$ and $f_{xx} > 0$ then we have a local minimum. If $D = 0$, we can not determine the nature of the critical point from the second derivatives alone. We also look already at **global maxima**, places where the function is larger or equal than the value at any other point.

Section 4.2: Lagrange Multipliers



Joseph Louis Lagrange

We extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to a system of equations which are called the **Lagrange equations**. They are a system of nonlinear equations $\nabla f = \lambda \nabla g, g = 0$. Extrema solve this equation of $\nabla g = 0$. When extremizing functions on a domain bounded by a curve $g(x, y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary. The second problem is a Lagrange problem. With the same method we can also extremize functions $f(x, y, z)$ of three variables, under the constraint $g(x, y) = 0, h(x, y) = 0$. In two or three dimensions, extrema could also be obtained without Lagrange from $\nabla f \times \nabla g = \vec{0}$.

Section 4.3: Double integrals



Guido Fubini

Integration in two dimensions is first done on **rectangles**, then on regions G bound by graphs of functions. Depending on whether curves $y = c(x), y = d(x)$ or curves $x = a(y), y = b(y)$ are the boundaries, we call the region **Type I** or **Type II**. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to derive results like **Fubini's theorem** on a rectangular region or the change of the order of integration which often enables the integration. The double integral $\iint_G f(x, y) \, dx dy$ is the signed volume under the graph of the function of two variables. Double integrals define **area** if $f(x, y) = 1$. By changing of order of integration in regions which are both type I and type II, we sometimes can integrate otherwise impossible integrals.

Section 4.4: Polar Integration

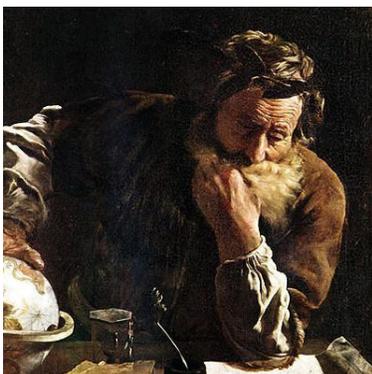


Bonaventura Cavalieri

Some regions can be described better in **polar coordinates** (r, θ) , where r is the distance to the origin and θ is the angle with the positive x -axes. Examples of regions which can be treated like that are **polar region** is $r \leq g(\theta)$ which trace flower-like shapes in the plane. An other application of double integrals is **surface area**. We derive the formula $\iint_R |r_u \times r_v| \, dudv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the surface area can be computed in closed form, like triangles, parts of the sphere or cylinder or paraboloid. Polar integration can also help to find one dimensional integrals which otherwise would be difficult to obtain.

Chapter 5. Line integrals

Section 5.1: Triple Integrals



Archimedes of Syracuse

Triple integrals measure volume, moment of inertia or the center of mass of a solid. First introduced for cuboids, then to more general regions like regions sandwiched between the graphs of two functions of two variables. Applications are computations of **mass** $\iiint_E \delta(x, y, z) \, dx dy dz$, **moment of inertia** $\iiint_E (x^2 + y^2 + z^2) \, dx dy dz$, **center of mass**, $\iiint \langle x, y, z \rangle \, dV$ the **expectation** $E[X] = \iiint X(x, y, z) \, dV / \iiint \, dV$ of a random variable $X(x, y, z)$ on a region Ω .

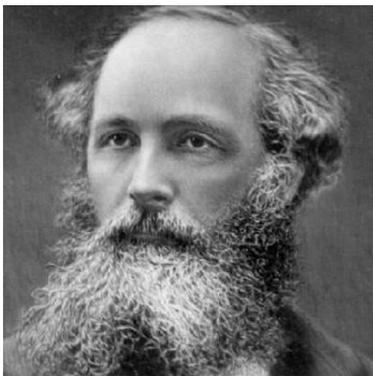
Section 5.2: Spherical Integration



Bernhard Riemann

Some objects can be described better in **cylindrical coordinates** (r, θ, z) , polar coordinates for the x, y variables in space. Examples of such regions are parts of cylinders or solids of revolution. The important factor to include when changing to cylindrical coordinates is r . Other regions are integrated over better in **spherical coordinates** (ρ, ϕ, θ) . Example of such regions are parts of cones or spheres. The important factor to include when changing to spherical coordinates is $\rho^2 \sin(\phi)$. As an application we compute **moments of inertia** of some bodies.

Section 5.3: Vector Fields



James Maxwell

Vector fields occur as force fields or velocity fields or mechanics. We look at examples in 2 or 3 dimensions. We learn how to **match vector fields** with formulas and introduce **flow lines**, parametrized curves $\vec{r}(t)$ for which the vector $\vec{F}(\vec{r}(t))$ is parallel to $\vec{r}'(t)$ at all times. Given a parametrized curve $\vec{r}(t)$ and a vector field \vec{F} , we can define the **line integral** $\int_C \vec{F}(\vec{r}(t)) \vec{r}'(t) dt$ along a curve in the presence of a vector field. An important example is the case if \vec{F} is a **force field** and where line integral is **work**.

Section 5.4: Line Integrals



André-Marie Ampère

For **conservative vector fields** one can evaluate a line integral using the **fundamental theorem of line integrals**. **Conservative** is also called **path independence** or **conservative** or being a **gradient field** $\vec{F} = \nabla f$. It is equivalent to being **irrotational** $\text{curl}(F) = Q_x - P_y = 0$ if the topological condition of **simply connected** is satisfied: any closed curve can be contracted to a point. The plane with the unit disc cut out is not simply connected because the path $\langle 2 \cos(t), 2 \sin(t) \rangle$ can not be pulled together to a point. In two dimensions, the curl of a field $\text{curl}\langle P, Q \rangle = Q_x - P_y$ measures the **vorticity** of the field and if this is zero, the line integral along a simply connected region is zero.

Chapter 6. Integral theorems

Section 6.1: Green's Theorem



Mikhail Ostrogradsky

Greens theorem equates the line integral along a boundary curve C with a double integral of the curl inside the region G : $\int \int_G \text{curl}(\vec{F})(x, y) dx dy = \int_C \vec{F}(\vec{r}(t)) \vec{r}'(t) dt$. The theorem is useful to **compute areas**: take a field $\vec{F} = \langle 0, x \rangle$ which has constant curl 1. It also allows to compute complicated line integrals. Greens theorem implies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and is therefore conservative. The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components of $\text{curl}(F)$ are the vorticity of the vector field in the x, y and z direction.

Section 6.2: Flux Integrals



Siméon Denis Poisson

Given a surface S and a fluid which has the velocity $\vec{F}(x, y, z)$ at the point (x, y, z) . the amount of fluid which passes through the membrane S in unit time is the **flux**. This integral is $\int \int_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dudv$. The angle between \vec{F} and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ determines the sign of $d\vec{S} = \vec{F} \cdot \vec{n} dudv$. Many concepts already come together in this definition: the parametrization of surfaces, the dot and cross product, as well as double integrals. We discuss how the derivatives **div**, **grad** and **curl** fit together. In one dimension, there is only one derivative, in two dimensions, there are two derivatives grad and curl and in three dimensions, there are three derivatives grad, curl and div.

Section 6.3: Stokes Theorem



George Gabriel Stokes

Stokes theorem equates the line integral along the boundary C of the surface with the flux of the "curl" of the field through the surface: $\int_C \vec{F} dr = \int \int_S \text{curl}(F) d\vec{S}$. The correct orientation is important. The theorem illustrates the Maxwell equations in electromagnetism and explains why the line integral of an irrotational field along a closed curve in space is zero if the region where \vec{F} is defined is simply connected. It is the flux of the curl of \vec{F} through the surface S bound by the curve C . In this lecture the Mathematica project is due. The project is creative, and tie connections with art.

Section 6.4 Divergence Theorem



Carl Friedrich Gauss

The total **divergence** of a vector field $\vec{F} = \langle P, Q, R \rangle$ inside solid E is the flux of \vec{F} through the boundary S . This **divergence theorem** equates the "local expansion rate" integrated over the solid $\iiint_E \operatorname{div}(\vec{F}) dV$ of a vector field \vec{F} with the flux $\iint_S \vec{F} \cdot d\vec{S}$ through the boundary surface S of E . Overview: In one dimension, there is one integral theorem, the **fundamental theorem of calculus**. In two dimensions, we have the **the fundamental theorem of line integrals in the plane** as well as **Greens theorem**. In three dimensions we have the **fundamental theorem of line integrals in space**, **Stokes theorem** and the **divergence theorem**. These integral theorems are of the form $\int_{\delta G} F = \int_G dF$, where δG is the boundary of G and dF is the derivative of F .