Name:

- Start by printing your name in the above box.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.

- Do not detach pages from this exam packet or unstaple the packet.

- Please try to write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids are allowed.

- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F — The lines $\vec{r_1}(t) = \langle t, t, -t \rangle$ and $\vec{r_2}(t) = \langle 1+t, 1+t, 1-t \rangle$ do not intersect.

2) T F — The quadratic surface $x^2 - y^2 = z^2$ is a hyperbolic paraboloid.

3) T F — If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}\,'(t) \neq 0$ everywhere, then they span a parallelepiped of volume 1.

4) T F — If $\vec{u} \cdot \vec{v} = 0$, then $\text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

5) T F — There is a vector field $\vec{F}(x,y)$ which has the property $\text{curl}(\vec{F}) = -\text{div}(\vec{F})$, where $\text{curl}(\vec{F})(x,y) = Q_x(x,y) - P_y(x,y)$ and $\text{div}(\vec{F})(x,y) = P_x(x,y) + Q_y(x,y)$.

6) T F — The acceleration vector $\vec{r}\,''(t) = \langle x(t), y(t) \rangle$ is always a unit vector if the velocity vector $\vec{r}\,'(t)$ is a unit vector.

7) T F — The grid curves $t \to \vec{r}(t, \phi)$ with fixed $0 < \phi < \pi$ for the standard parametrization of the unit sphere have curvature $1/\sin(\phi)$.

8) T F — Any smooth function $f(x,y)$ has a local maximum somewhere in the plane.

9) T F — The linearization $L(x,y)$ of constant function $f(x,y) = 3$ is $L(x,y) = 3$.

10) T F — A gradient field is incompressible: it satisfies $\text{div}(F) = 0$ everywhere.

11) T F — If $f(x,y)$ has a maximum under the constraint $g(x,y) = 1$, then $\nabla f = \langle 0, 0 \rangle$ at this point.

12) T F — Assume a vector field $\vec{F}(x,y,z)$ is the curl of a vector field $\vec{G}$ then the flux of the field F through the ellipsoid $x^2 + y^2 + 5z^2 \leq 1$ is zero.

13) T F — If the divergence of a field $\vec{F}$ are zero everywhere, then any line integral along a closed curve is zero.

14) T F — The gradient of the divergence of a field is always the zero field.

15) T F — The vector field $\vec{F}(x,y,z) = \langle x^2, y^2, z^3 \rangle$ is a gradient field.

16) T F — The volume of a solid can be computed as the flux of the field $\langle 0, y, 0 \rangle$ through the boundary surface.

17) T F — The curvature of a line is zero.

18) T F — The distance between the unit sphere centered at $(0,0,0)$ and the plane $z = 5$ is equal to 4.

19) T F — The partial differential equation $u_t = u_x$ is called heat equation.

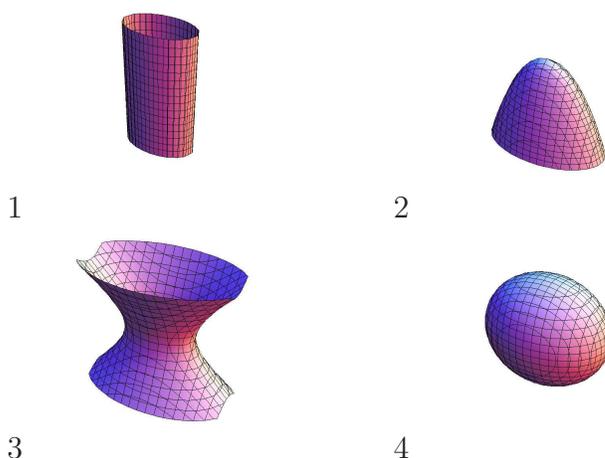20) T F — The point $(1, -1, \sqrt{2})$ in spherical coordinates is $(\rho, \phi, \theta) = (2, \pi/4, 3\pi/2)$.

Problem 2) (10 points) No justifications are necessary.
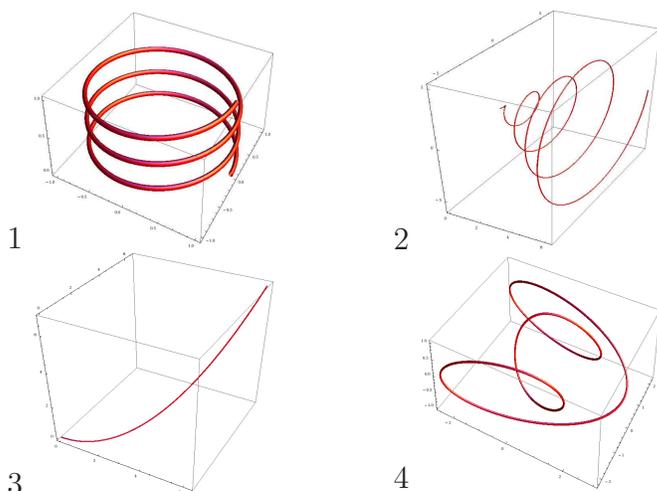
a) (4 points) Match the objects with the definitions.

1   2

3   4
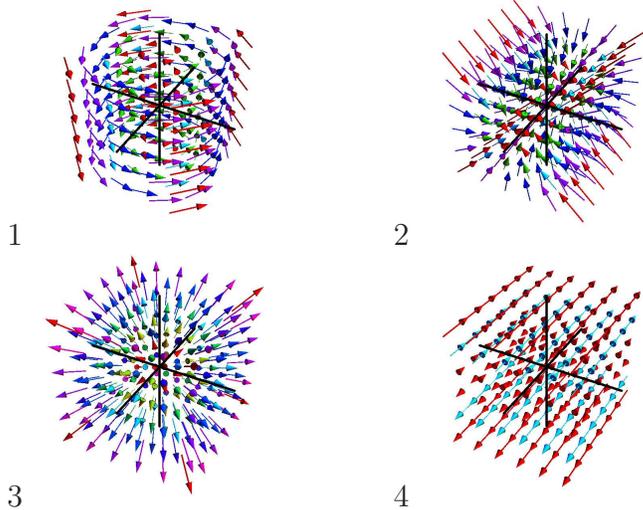
enter 1-4	vector field
	$\vec{F}(x,y,z) = \langle x, y, z \rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$
	$\vec{F}(x,y,z) = \langle 0, z, 0 \rangle$
	$\vec{F}(x,y,z) = \langle -x, 0, -z \rangle$

b) (3 points) Match the surfaces with their names: (put O if no match)

1   2

3   4

enter 1-4	surface
	$x^2 + y^2 + 3z = 0$
	$x^2 + y^2 - 3z^2 = 1$
	$x^2 + y^2 + 3z^2 = 1$
	$x^3 + 3y^2 = 1$

c) (3 points) Match the space curves

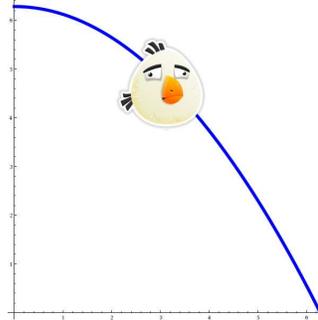1   2

3   4

1-4	parametrized curve
	$\vec{r}(t) = \langle t, t^2, t^3 \rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(3t), t \rangle$
	$\vec{r}(t) = \langle (2 + \cos(t)) \cos(3t),$ $(2 + \cos(t)) \sin(t),$ $\sin(3t) \rangle$
	$\vec{r}(t) = \langle t, t\cos(3t), t\sin(3t) \rangle$

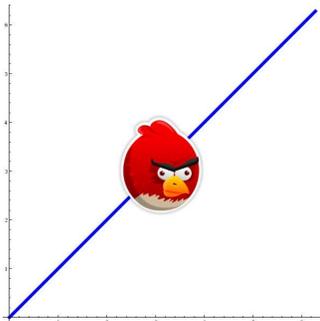Problem 3) (10 points) No justifications are necessary

a) (5 points) We watch "angry birds" attacking on curves with acceleration $\vec{r}\,''(t)$. (The pictures show the $xz-$ planes and the birds start with a constant velocity $\langle 1, 0, 0 \rangle$.) Match the displayed curves $\vec{r}(t)$ with the formulas for accelerations.
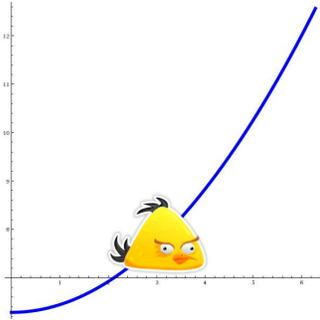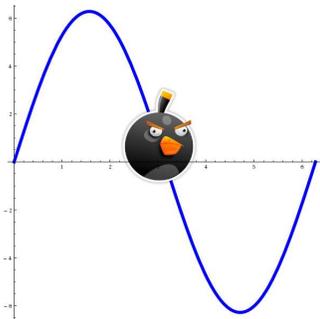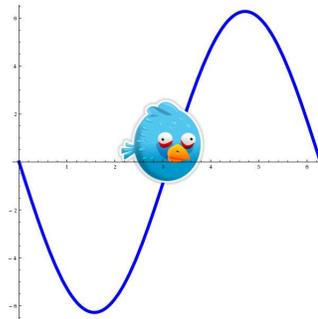

1

2

3

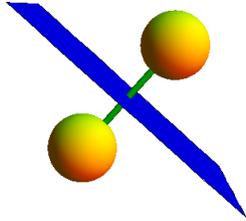| acceleration	enter curve 1-5
$\vec{r}\,''(t) = \langle 0, 0, \sin(t) \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, 10 \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, -\sin(t) \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, 0 \rangle$	

4

5

b) (5 points) Match the formulas: (put O if no match)

label	formula
A	$\vec{r}\,'(t)$
B	$\int_0^1 |\vec{r}\,'(t)|\ dt$
C	$\vec{r}\,'(t)/|\vec{r}\,'(t)|$
D	$\vec{T}\,'(t)/|\vec{T}\,'(t)|$
E	$|\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$

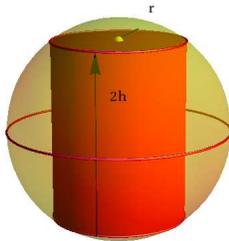| expression	enter A-E
Curvature	
Unit tangent vector	
Unit normal vector	
Velocity	
Arc length	

Problem 4) (10 points)

a) (5 points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y - 1)^2 + z^2 = 1$, $(x - 5)^2 + y^2 + z^2 = 1$.

b) (5 points) Find the plane perpendicular to the line L for which the distances to the spheres are the same.
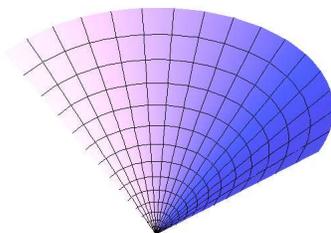
Problem 5) (10 points)

Johannes Kepler asked which cylinder or radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve his problem, use the Lagrange method and maximize the volume

$$f = 2\pi r^2 h$$

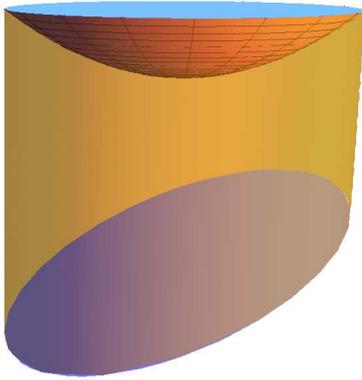under the constraint that $r^2 + h^2 = 1$.

Problem 6) (10 points)

a) (6 points) Find the surface area of the surface

$$r(u, v) = \langle v^2 \cos(u), v^2 \sin(u), v^2 \rangle, 0 \le u \le \pi, 0 \le v \le 1 .$$

b) (4 points) Find the arc length of the boundary curve $\vec{r}(u, 1)$ where $0 \le u \le \pi$.
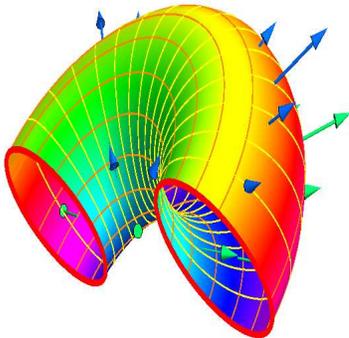
Problem 7) (10 points)

Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x,y) = x - y$ and $g(x,y) = x^2 + y^2 + 4$.

Problem 8) (10 points)

Find the flux of the curl of the vector field

$$\vec{F}(x,y,z) = \langle x, y, z + \sin(\sin(y^2)) \rangle$$

through the torus

$$\vec{r}(s,t) = \langle (2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s) \rangle$$

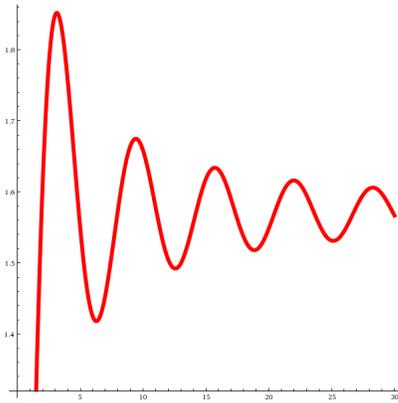with $0 \leq t \leq \pi$ and $0 \leq s < 2\pi$.

Problem 9) (10 points)

Heron's formula for the area A of a triangle of side length $x, y, 1$ satisfies $16A^2 = f(x,y)$, where

$$f(x,y) = -1 + 2x^2 - x^4 + 2y^2 + 2x^2y^2 - y^4 .$$

Classify all the critical points of f. Is there a global maximum of f and so for the area?

Remark not to worry about: The formula follows directly from Heron's formula $s = (a + b + 1)/2; A = \sqrt{s(s-a)(s-b)(s-1)}$.
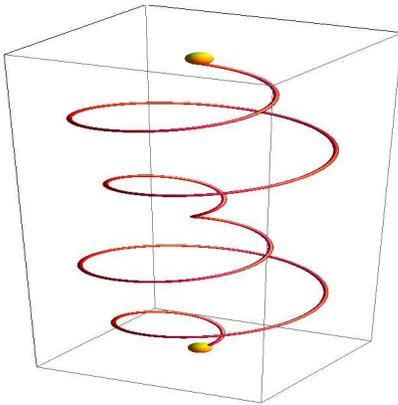
Problem 10) (10 points)

The anti derivative of the **sinc** function

$$\frac{\sin(x)}{x}$$

is called the **sine integral** $\text{Si}(x)$. It can not be expressed in terms of known functions. Still we can compute the following double integral

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} \, dy \, dx \ .$$
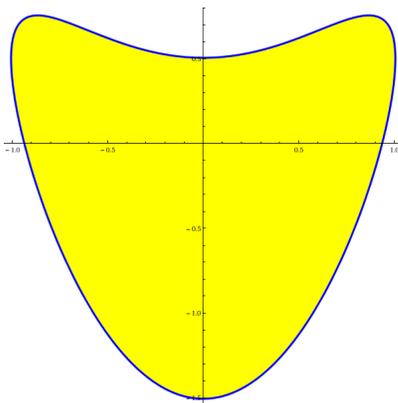
Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x, y, z) = \langle -x^{10}, \sin(y), z^3 \rangle$$

along the curve $\vec{r}(t) = \langle \sin(t)\cos(5t), \sin(t)\sin(5t), t \rangle$ where $0 \le t \le 2\pi$.
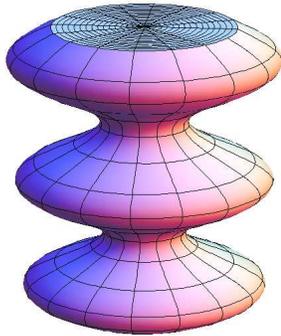
Problem 12) (10 points)

Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t) + \cos(2t)/2 \rangle \ ,$$

where $0 \le t < 2\pi$.

Find the flux of the vector field

$$\vec{F}(x, y, z) = \langle x^3/3, y^3/3, \sin(xy^5) \rangle$$

through the boundary surface of the solid bound by the surface of revolution $\vec{r}(t, z) = \langle (2 + \sin(z)) \cos(t), (2 + \sin(z)) \sin(t), z \rangle$ and the planes $z = 0$, $z = 3$. The surface is oriented so that the normal vector points outwards.