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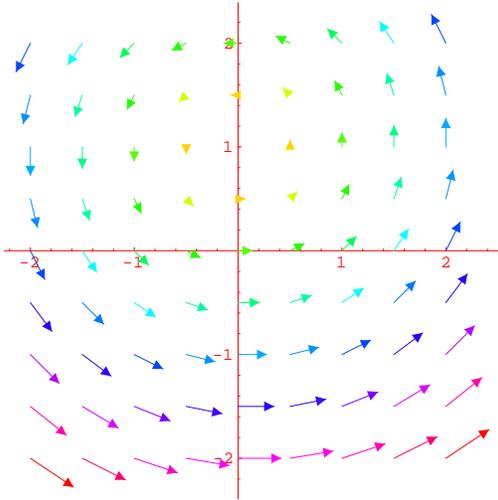
- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

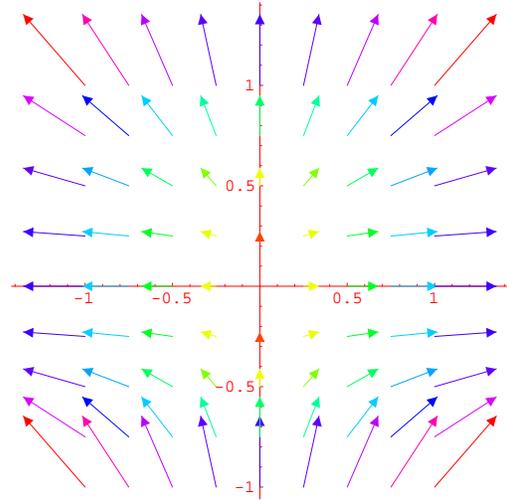
Problem 1) (20 points)

- 1)  T  F For any two nonzero vectors  $\vec{v}, \vec{w}$  the vector  $\vec{v} - \vec{w}$  is perpendicular to  $\vec{v} \times \vec{w}$ .
- 2)  T  F The cross product satisfies the law  $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$ .
- 3)  T  F If the curvature of a smooth curve  $\vec{r}(t)$  in space is defined and zero for all  $t$ , then the curve is part of a line.
- 4)  T  F The curve  $\vec{r}(t) = (1 - t)A + tB, t \in [0, 1]$  connects the point  $A$  with the point  $B$ .
- 5)  T  F For every  $c$ , the function  $u(x, t) = (2 \cos(ct) + 3 \sin(ct)) \sin(x)$  is a solution to the wave equation  $u_{tt} = c^2 u_{xx}$ .
- 6)  T  F The arc length of  $\vec{r}(t) = (t, \sin(t)), t \in [0, 2\pi]$  is  $\int_0^{2\pi} \sqrt{1 + \cos^2(t)} dt$ .
- 7)  T  F Let  $(x_0, y_0)$  be the maximum of  $f(x, y)$  under the constraint  $g(x, y) = 1$ . Then  $f_{xx}(x_0, y_0) < 0$ .
- 8)  T  F The function  $f(x, y, z) = x^2 - y^2 - z^2$  decreases in the direction  $(2, -2, -2)/\sqrt{12}$  at the point  $(1, 1, 1)$ .
- 9)  T  F  $\vec{F}$  is a vector field for which  $|\vec{F}(x, y, z)| \leq 1$ . For every curve  $C : \vec{r}(t)$  with  $t \in [0, 1]$ , the line integral  $\int_C \vec{F} \cdot d\vec{r}$  is  $\leq$  the arc length of  $C$ .
- 10)  T  F Let  $\vec{F}$  be a vector field and  $C$  is a curve which is a flow line, then  $\int_C \vec{F} \cdot d\vec{r} > 0$ .
- 11)  T  F The divergence of the gradient of any  $f(x, y, z)$  is always zero.
- 12)  T  F For every function  $f$ , one has  $\text{div}(\text{curl}(\text{grad}(f))) = 0$ .
- 13)  T  F If for two vector fields  $\vec{F}$  and  $\vec{G}$  one has  $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$ , then  $\vec{F} = \vec{G} + (a, b, c)$ , where  $a, b, c$  are constants.
- 14)  T  F For every vector field  $\vec{F}$  the identity  $\text{grad}(\text{div}(\vec{F})) = \vec{0}$  holds.
- 15)  T  F If a nonempty quadric surface  $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$  can be contained inside a finite box, then  $a, b, c \geq 0$ .
- 16)  T  F If  $\vec{F}$  is a vector field in space then the flux of  $\vec{F}$  through any closed surface  $S$  is 0.
- 17)  T  F If  $\text{div}(\vec{F})(x, y, z) = 0$  for all  $(x, y, z)$ , then  $\text{curl}(\vec{F}) = (0, 0, 0)$  for all  $(x, y, z)$ .
- 18)  T  F The flux of the vector field  $\vec{F}(x, y, z) = (y + z, y, -z)$  through the boundary of a solid region  $E$  is equal to the volume of  $E$ .
- 19)  T  F If in spherical coordinates the equation  $\phi = \alpha$  (with a constant  $\alpha$ ) defines a plane, then  $\alpha = \pi/2$ .
- 20)  T  F For every function  $f(x, y, z)$ , there exists a vector field  $\vec{F}$  such that  $\text{div}(\vec{F}) = f$ .

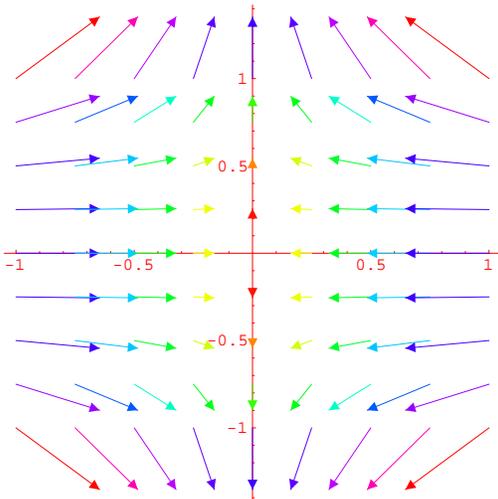
Problem 2) (10 points)



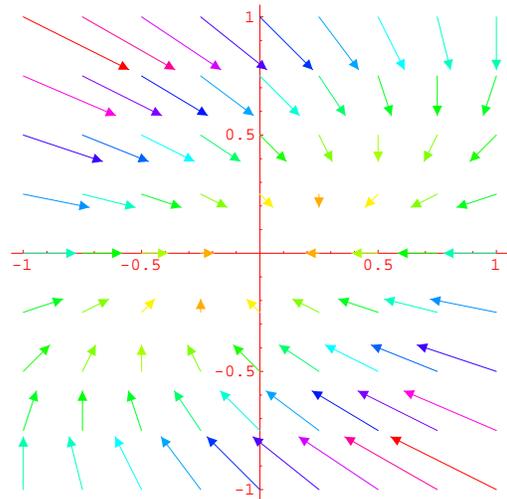
I



II



III



IV

For the sign of the curl or divergence, where either + (positive), - (negative) or 0 for zero. The vector fields are considered on the square  $[-1/2, 1/2] \times [-1/2, 1/2]$  in this problem.

Enter I,II,III,IV here	Vector field	curl sign	divergence sign
	$F(x, y) = (x, y^2)$		
	$F(x, y) = (1 - y, x)$		
	$F(x, y) = (y - x, -y)$		
	$F(x, y) = (-x, y^3)$		

Problem 3) (10 points)

Mark with a cross in the column below "conservative" if a vector fields is conservative (that is if  $\text{curl}(\vec{F})(x, y, z) = (0, 0, 0)$  for all points  $(x, y, z)$ ). Similarly, mark the fields which are incompressible (that is if  $\text{div}(\vec{F})(x, y, z) = 0$  for all  $(x, y, z)$ ). No justifications are needed.

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$		
$\vec{F}(x, y, z) = (x, y, z)$		
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$		

Problem 4) (10 points)

Let  $E$  be a parallelogram in three dimensional space defined by two vectors  $\vec{u}$  and  $\vec{v}$ .

- (3 points) Express the diagonals of the parallelogram as vectors in terms of  $\vec{u}$  and  $\vec{v}$ .
- (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

c) (4 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid  $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ .

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} dx dy.$$

Problem 7) (10 points)

Evaluate  $\iint_D \frac{2xy}{x^2+y^2} dx dy$ , where  $D$  is the intersection of the annulus  $1 \leq x^2 + y^2 \leq 2$  with the second quadrant  $\{x \leq 0, y \geq 0\}$ .

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function  $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$ .

b) (3 points) Classify the critical points.

c) (2 points) Locate the local and absolute maxima of  $f$ .

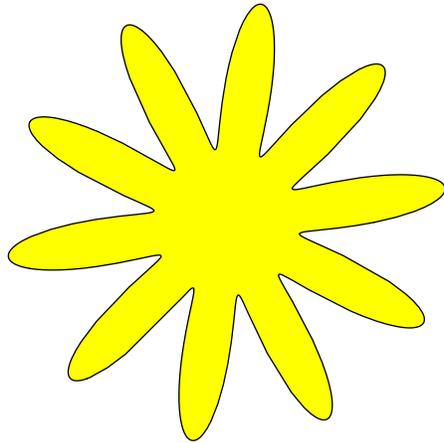
d) (2 points) Find the equation for the tangent plane to the graph of  $f$  at each absolute maximum.

Problem 9) (10 points)

Find the area  $\iint_R 1 \, dx dy$  of the 10 legged "sea star"  $R$ , enclosed by the polar curve

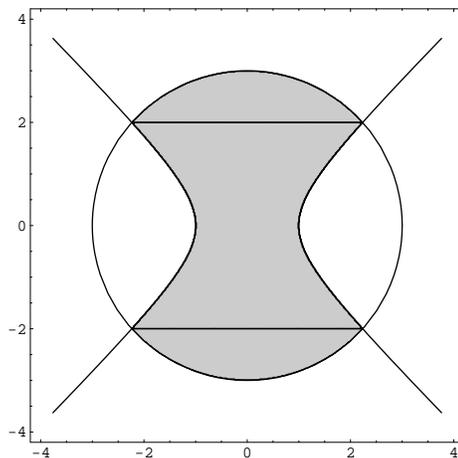
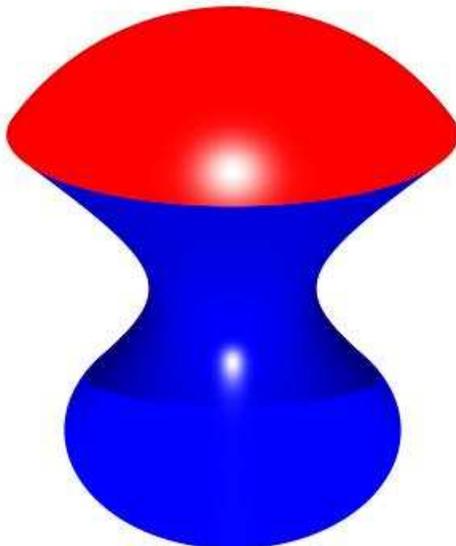
$$r(\theta) = 2 + \sin(10\theta),$$

where  $\theta \in [0, 2\pi]$ . The photo to the right shows a real sea star.



Problem 10) (10 points)

Find the volume of the intersection of the interior of the one sided hyperboloid  $x^2 + y^2 - z^2 \leq 1$  with the solid ball enclosed by the sphere  $x^2 + y^2 + z^2 \leq 9$ .



Problem 11) (10 points)

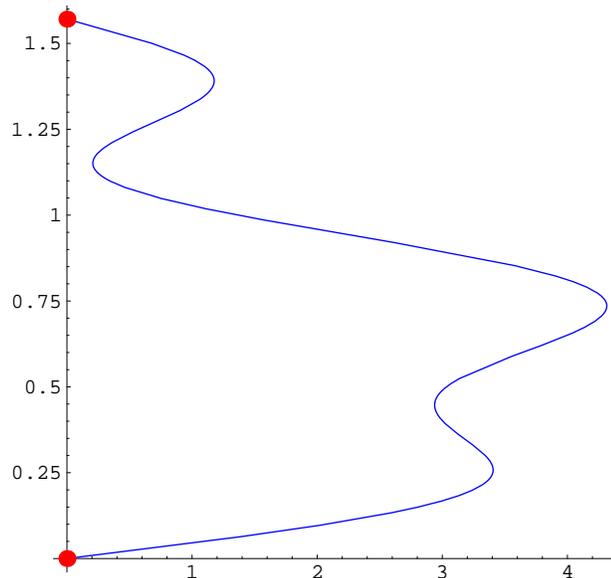
Let the curve  $C$  be parametrized by  $\vec{r}(t) = (t, \sin t, t^2 \cos t)$  for  $0 \leq t \leq \pi$ . Let  $f(x, y, z) = z^2 e^{x+2y} + x^2$  and  $\vec{F} = \nabla f$ . Find  $\int_C \vec{F} \cdot d\vec{r}$ .

Problem 12) (10 points)

- a) Find the linear approximation  $L(x, y)$  of  $f(x, y) = \sqrt{4 + 2x^2 + 4y^2}$  at the point  $(x, y) = (2, 1)$ .  
b) Find the equation for the tangent line to the level curve of  $f(u, v)$  at  $(2, 1)$ .

Problem 13) (10 points)

Find the line integral of the vector field  $\vec{F}(x, y) = \langle x^{30} + y, y^{50} + x \rangle$  along the path  $\vec{r}(t) = \langle 4 \sin(\pi \sin(t)) + \sin(10t), t \rangle$  with  $0 \leq t \leq \pi/2$ .



Problem 14) (10 points)

Evaluate the line integral of the vector field  $\vec{F}(x, y) = (y^2, x^2)$  in the clockwise direction around the triangle in the  $xy$ -plane defined by the points  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$  in two ways:

- a) (5 points) by evaluating the three line integrals.
- b) (5 points) using Greens theorem.

Problem 15) (10 points)
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Use Stokes theorem to evaluate the line integral of  $\vec{F}(x, y, z) = (-y^3, x^3, -z^3)$  along the curve  $\vec{r}(t) = (\cos(t), \sin(t), 1 - \cos(t) - \sin(t))$  with  $t \in [0, 2\pi]$ .