8/8/2013 PRACTICE EXAM VIII **Maths 21a, O. Knill, Summer 2013**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

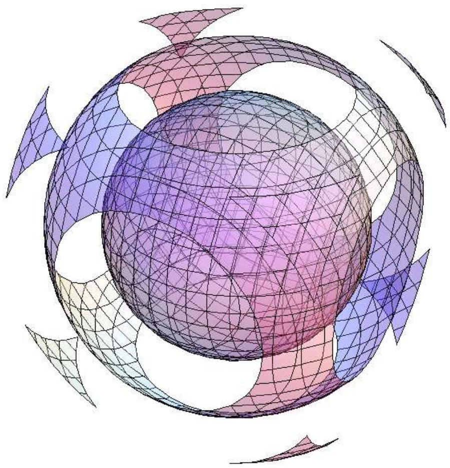
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points)

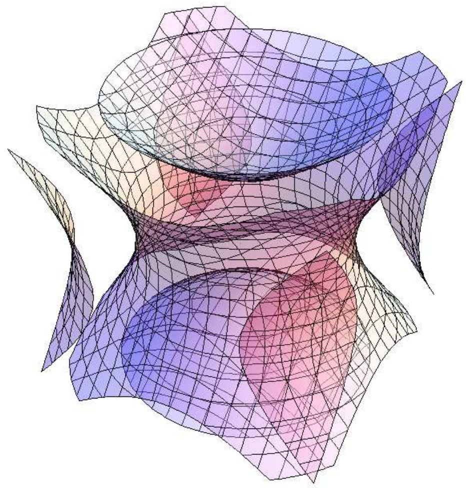
1)	T	F	The quadratic surface $-x^2 + y^2 + z^2 = -5$ is a one-sheeted hyperboloid.
2)	T	F	$\lvert\vec{u}\times\vec{v}\rvert < \lvert\vec{u}\cdot\vec{v}\rvert$ implies that the angle α between u and v satisfies $\lvert\alpha\rvert < \pi/4$.
3)	T	F	$\int_0^3 \int_0^{2\pi} r\sin(\theta)\, d\theta\, dr$ is the area of a disc of radius 3.
4)	T	F	If a vector field $\vec{F}(x,y,z)$ satisfies $\mathrm{curl}(F)(x,y,z) = 0$ for all points (x,y,z) in space, then $\vec{F}$ is a gradient field.
5)	T	F	The acceleration of a parameterized curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is zero if the curve $\vec{r}(t)$ is a line.
6)	T	F	The curvature of the curve $\vec{r}(t) = \langle 3+\sin(t), t, t^2\rangle$ is half of the curvature of the curve $\vec{s}(t) = \langle 6+2\sin(t), 2t, 2t^2\rangle$.
7)	T	F	The curve $\vec{r}(t) = \langle \sin(t), t, \cos(t)\rangle$ for $t \in [0,\pi]$ is a helix.
8)	T	F	If a function $u(t,x)$ is a solution of the partial differential equation $u_{tx} = 0$, then it is constant.
9)	T	F	The unit tangent vector $\vec{T}$ of a curve is always perpendicular to the acceleration vector.
10)	T	F	Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then the gradient of g at (x_0,y_0) is perpendicular to the gradient of f at (x_0,y_0).
11)	T	F	At a critical point for which $f_{xx} > 0$, the discriminant D determines whether the point is a local maximum or a local minimum.
12)	T	F	If a vector field $\vec{F}(x,y)$ is a gradient field, then for any closed curve C the line integral $\int_C \vec{F}\cdot d\vec{r}$ is zero.
13)	T	F	If C is part of a level curve of a function $f(x,y)$ and $\vec{F} = \langle f_x, f_y\rangle$ is the gradient field of f, then $\int_C \vec{F}\cdot d\vec{r} = 0$.
14)	T	F	The gradient of the divergence of a vector field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is always the zero vector field.
15)	T	F	The line integral of the vector field $\vec{F}(x,y,z) = \langle x,y,z\rangle$ along a line segment from $(0,0,0)$ to $(1,1,1)$ is 1.
16)	T	F	If $\vec{F}(x,y) = \langle x^2 - y, x\rangle$ and $C: \vec{r}(t) = \langle \sqrt{\cos(t)}, \sqrt{\sin(t)}\rangle$ parameterizes the boundary of the region $R: x^4 + y^4 \leq 1$, then $\int_C \vec{F}\cdot d\vec{r}$ is twice the area of R.
17)	T	F	The flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ through the boundary S of a solid torus E is equal to the volume the torus.
18)	T	F	If $\vec{F}$ is a vector field in space and S is the boundary of a cube then the flux of $\vec{\mathrm{curl}}(\vec{F})$ through S is 0.
19)	T	F	If $\mathrm{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) and S is a half sphere then the flux of $\vec{F}$ through S is zero.
20)	T	F	If the curl of a vector field is zero everywhere, then its divergence is zero everywhere too.

Problem 2) (10 points)

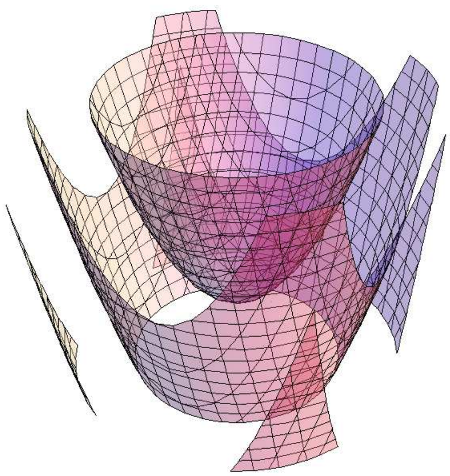
a) Match the following contour surface maps with the functions $f(x, y, z)$



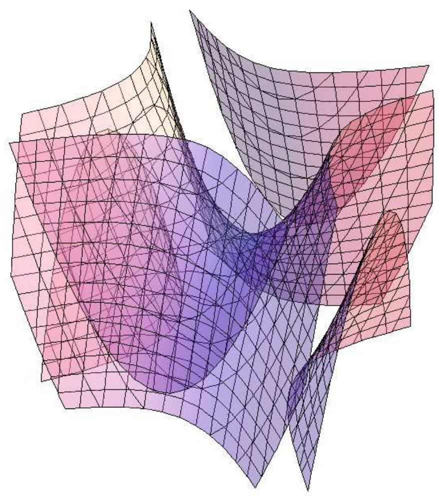
I



II



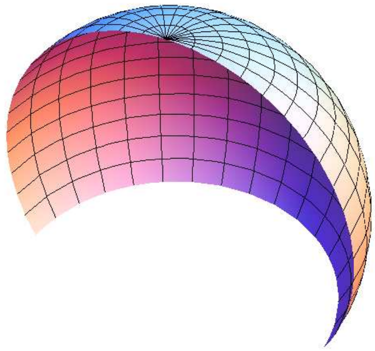
III



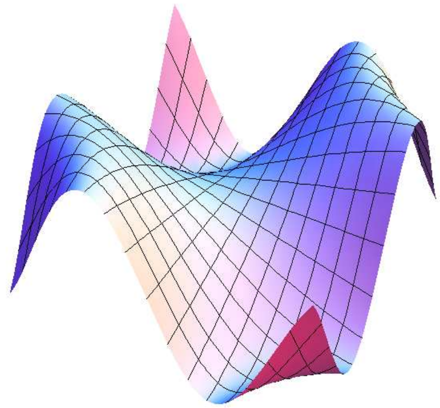
IV

Enter I,II,III,IV here	Function
	$f(x, y, z) = -x^2 + y^2 + z$
	$f(x, y, z) = x^2 + y^2 + z^2$
	$f(x, y, z) = -x^2 - y^2 + z$
	$f(x, y, z) = -x^2 - y^2 + z^2$

b) Match the following parametrized surfaces with their definitions



I



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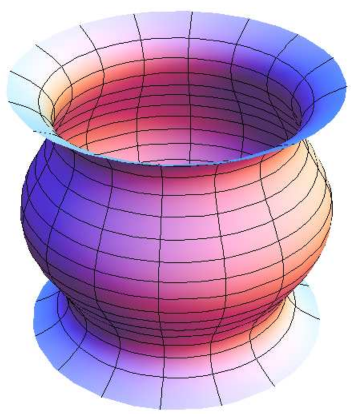

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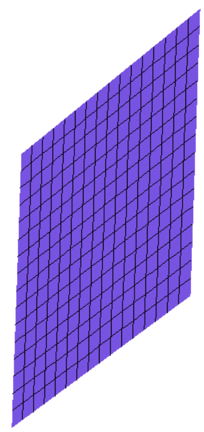
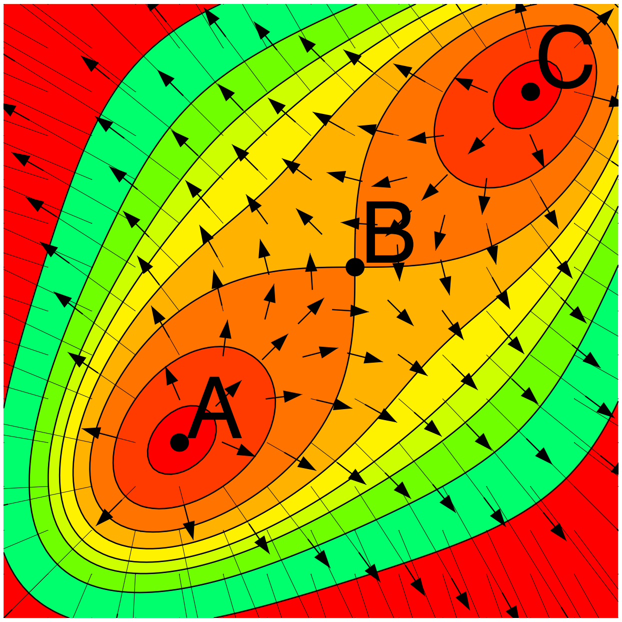


IV

Enter I,II,III,IV here	Function
	$\vec{r}(u,v) = \langle u - v, u + 2v, 2u + 3v\sin(uv)\rangle$
	$\vec{r}(u,v) = \langle \cos(u)\sin(v), 4\sin(u)\sin(v), 3\cos(v)\rangle$
	$\vec{r}(u,v) = \langle (v^4 - v^2 + 1)\cos(u), (v^4 - v^2 + 1)\sin(u), v\rangle$
	$\vec{r}(u,v) = \langle u, v, \sin(uv)\rangle$

Problem 3) (10 points)

No justifications are required in this problem. The first picture shows a gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$.



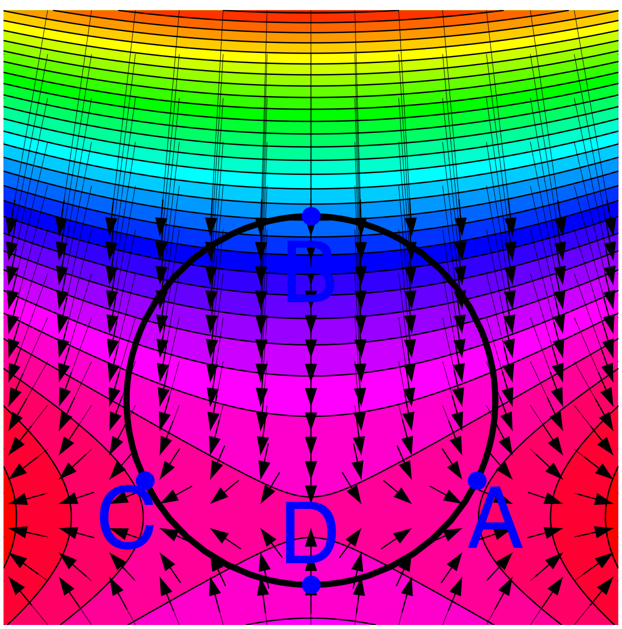
The critical points of $f(x,y)$ are called A, B and C. What can you say about the nature of these three critical points? Which one is a local max, which a local min, which a saddle.

point	local max	local min	saddle
A			
B			
C			

$\int_{P\to Q} \vec{F} \cdot \vec{dr}$ denotes the line integral of $\vec{F}$ along a straight line path from P to Q.

statement	True	False
$\int_{A\to B} \vec{F}\ \vec{dr} \geq 0$		
$\int_{A\to C} \vec{F}\ \vec{dr} \geq \int_{A\to B} \vec{F}\ \vec{dr}$		

The second picture again shows an other gradient vector field $\vec{F} = \nabla f(x,y)$ of a different function $f(x,y)$.



We want to identify the maximum of $f(x,y)$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. The solutions of the Lagrange equations in this case are labeled A, B, C, D. At which point on the circle if f maximal?

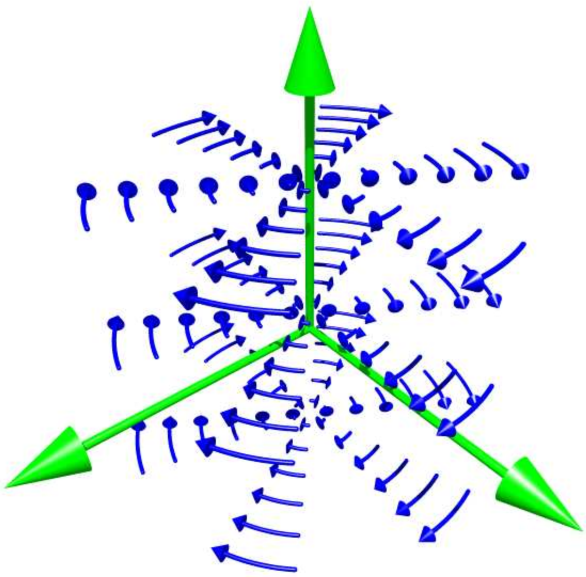
point	maximum
A	
B	
C	
D	

$\int_\gamma \vec{F} \cdot \vec{dr}$ denotes the line integral of $\vec{F}$ along the circle $\gamma : x^2 + y^2 = 1$, oriented counter clockwise.

$\int_\gamma \vec{F} \cdot \vec{r}$	> 0	< 0	$= 0$
Check if true:			

Problem 4) (10 points)

a) Match the following vector fields in space with the corresponding formulas:



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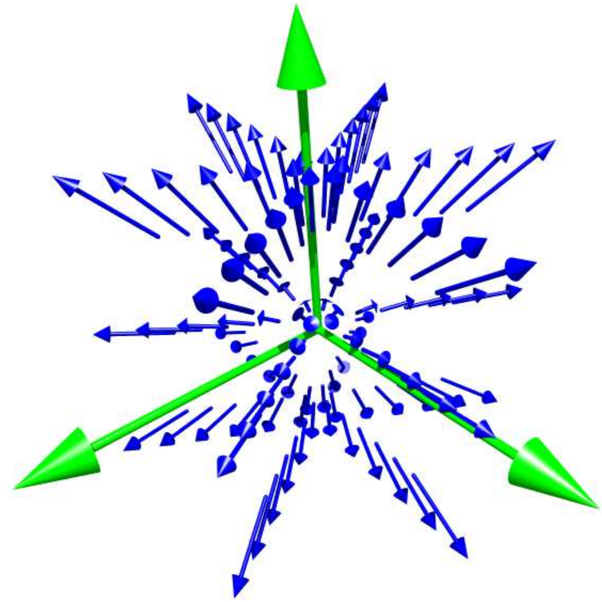

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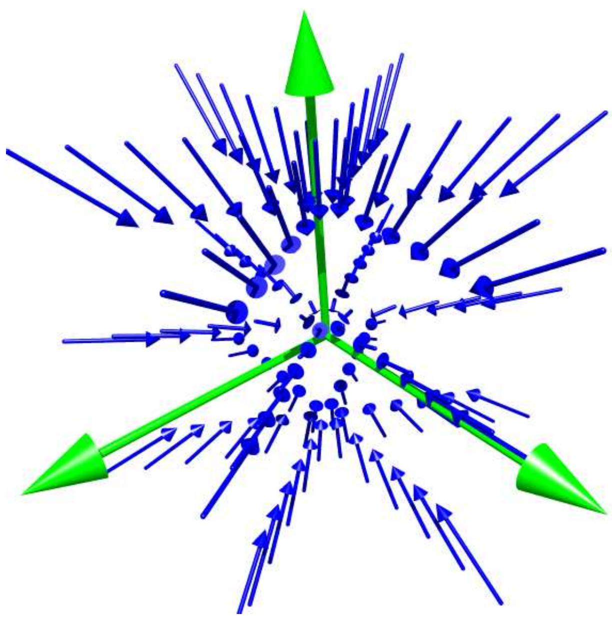
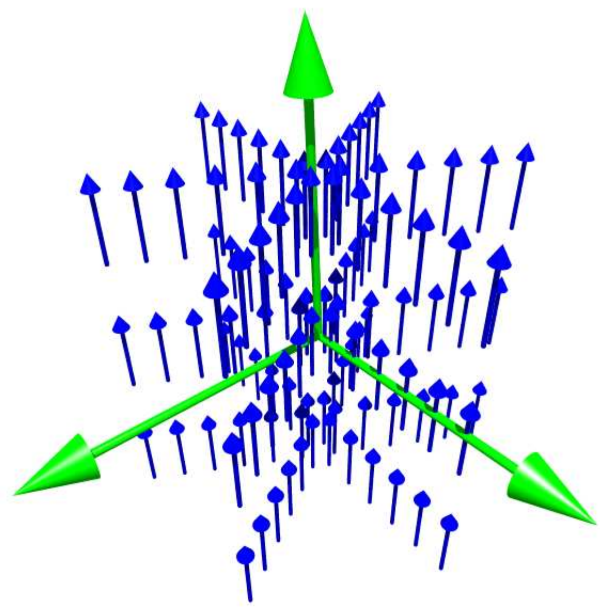


III



IV

Enter I,II,III,IV here	Vector Field
	$\vec{F}(x, y, z) = \langle y, -x, 0\rangle$
	$\vec{F}(x, y, z) = \langle x, y, z\rangle$
	$\vec{F}(x, y, z) = \langle 0, 0, 1\rangle$
	$\vec{F}(x, y, z) = \langle -x, -2y, -z\rangle$

b) Choose from the following words to complete the following table: "arc length formula", "surface area formula", "chain rule", "volume of parallel epiped", "area of parallelogram", "Consequence of Clairot theorem", "Fubini Theorem", "line integral", "Flux integral", "vector projection", "scalar projection", "partial derivative":

Formula	Name of formula or rule or theorem
$P_{\vec{v}}(\vec{w}) = \vec{w}\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}$	
$\int\int_R \|\vec{r}_u \times \vec{r}_v\| \, dudv$	
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$	
$\text{curl}(\text{grad}(f)) = \vec{0}$	
$\int_a^b \|\vec{r}\,'(t)\| \, dt$	
$\|\vec{u} \cdot (\vec{v} \times \vec{w})\|$	

Problem 5) (10 points)

The point $P = (1,2,2)$ is mirrored at the plane which contains the points $A = (0,0,0)$, $B = (4,0,2)$ and $C = (2,2,-1)$. The mirror point is called Q.

a) (3 point) Find a normal vector to the plane which has length 1.

b) (4 points) Find the distance of the point P to the plane.

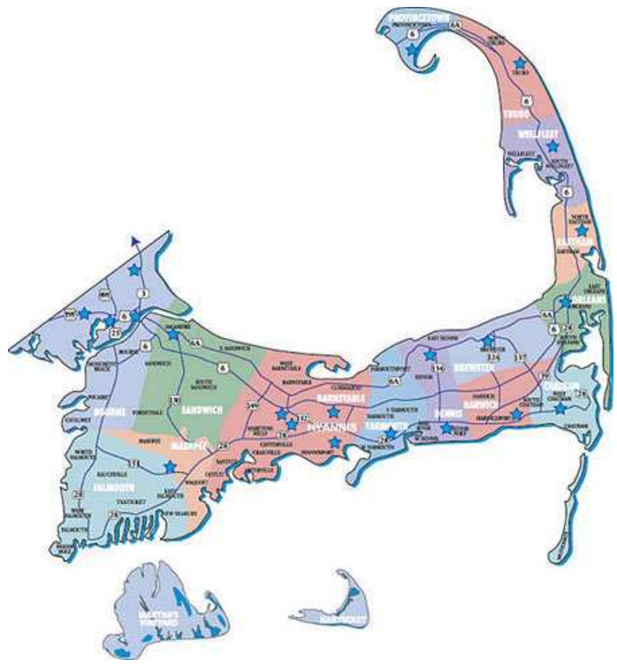
c) (3 points) Find the coordinates of the point Q.

Problem 6) (10 points)

The tip of Cape Cod near Provincetown is full of dunes and a nice place to relax. The elevation at position (x, y) is

$$f(x,y) = x^3/12 - xy - 2y - y^2 - x \; .$$

Find all critical points and decide whether they are maxima, minima or saddle points.

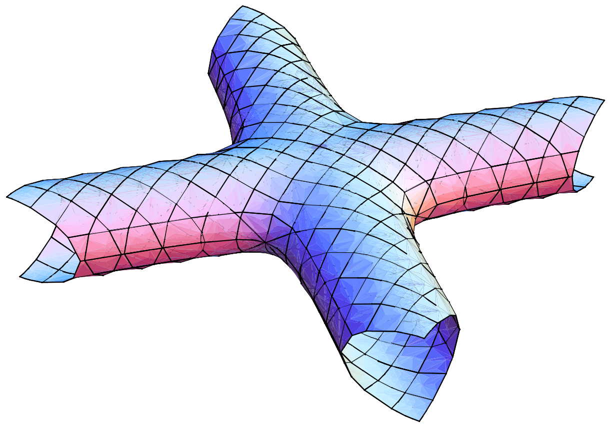


Problem 7) (10 points)

The **roman surface** is defined as the implicit surface

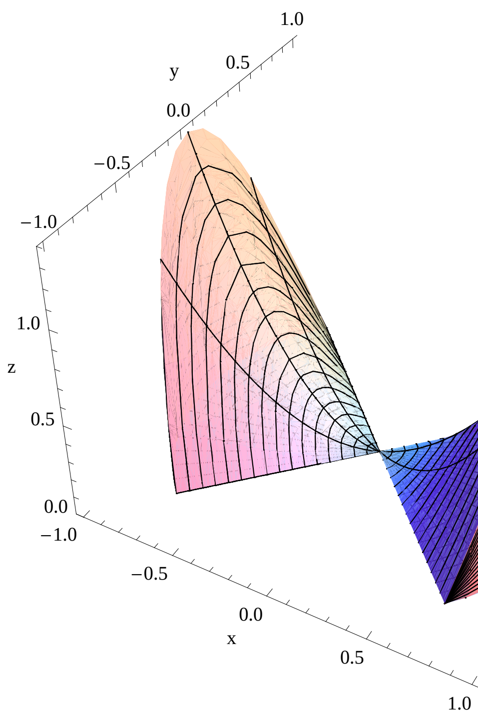
$$f(x, y, z) = x^4 - 2x^2y^2 + y^4 + 4x^2z + 4y^2z + 4x^2z^2 + 4y^2z^2 = 0 \ .$$

a) (4 points) Find a vector normal to the surface at the point $P = (-1, 1, -1)$.
b) (3 points) Find the equation of the tangent plane at P.
c) (3 points) Estimate $f(-1, 1, -1.1)$ by linear approximation.



Problem 8) (10 points)

An excentric architect builts a cinema, which is above the xy-plane and below the surface $z = x^2 - y^2$ and within the solid cylinder $x^2 + y^2 \leq 1$. Find the volume of this building, which features two movie presentation halls.

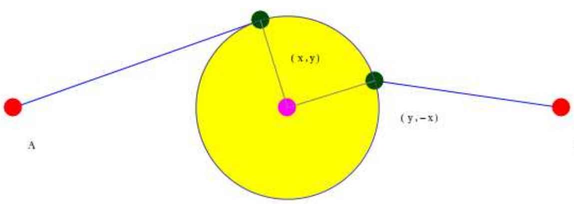


Problem 9) (10 points)

A wheel of radius 1 is attached by rubber bands with two points $(-3, 0)$ and $(3, 0)$. The point (x, y) is attached to $(-3, 0)$ and $(y, -x)$ is attached to $(3, 0)$. The point (x, y) is constrained to $g(x, y) = x^2 + y^2 = 1$. The wheel will settle at the position, for which the potential energy of the wheel

$$f(x, y) = (x + 3)^2 + y^2 + (y - 3)^2 + x^2$$

is minimal. Find that position.



Problem 10) (10 points)

Find the surface area of the surface

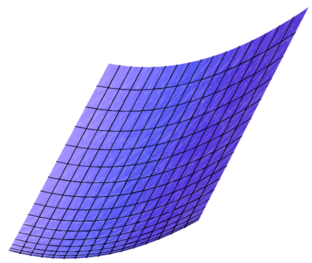
$$\vec{r}(u, v) = \langle u, v^2, \frac{u^2}{\sqrt{2}} + v^2 \rangle \ ,$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

Hint. In class we have established

$$\int \sqrt{1 + x^2} \ dx = x\sqrt{1 + x^2}/2 + \text{arcsinh}(x)/2$$

which you can use without having to rederive it.



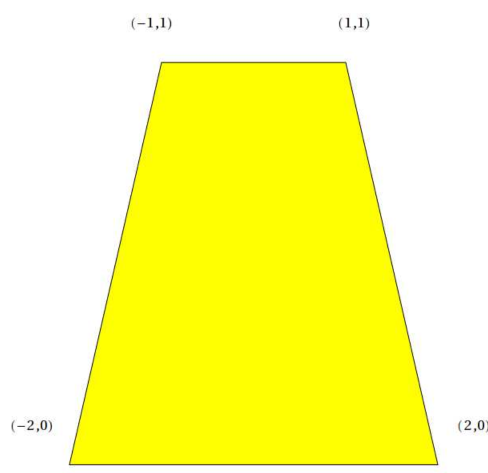
Problem 11) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{\sqrt{4-\sqrt{x}}}\, dx\, dy \; .$$

Problem 12) (10 points)

Find the line integral of the vector field $\vec{F}(x,y) = \langle 3y, 10x + \log(10+\sqrt{y})\rangle$ along the boundary C of the trapezoid with vertices $(-2,0),(2,0),(1,1),(-1,1)$.



Problem 13) (10 points)

Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = \langle x+z, y+x, z+y+\cos(\sin(\cos(z)))\rangle$$

and let S be the triangle with vertices $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$ parametrized by $\vec{r}(s, t) = \langle 1, 0, 0\rangle + t\langle -1, 1, 0\rangle + s\langle -1, 0, 1\rangle$. Find the line integral along the curve $A \to B \to C \to A$, which is the boundary of S.

Problem 14) (10 points)

What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle 33x + \cos(y^2 \log(1 + y^2)), x^{9999}, \sqrt{2 + y^2 + \sin(\sin(xy^7))}\rangle$$

through the boundary S of the cone $E = \{x^2 + y^2 \leq z^2,\ 0 \leq z \leq 1\ \}$. All boundary parts of the cone are oriented so that the normal vector points outwards.