Lecture 13: Extrema

An important problem in multi-variable calculus is to **extremize** a function $f(x, y)$ of two variables. As in one dimensions, in order to look for maxima or minima, we consider points, where the "derivative" is zero.

A point (a, b) in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$.

Critical points are candidates for extrema because at critical points, all directional derivatives $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ are zero. We can not increase the value of f by moving into any direction.

This definition does not include points, where f or its derivative is not defined. We usually assume that a function is arbitrarily often differentiable. Points where the function has no derivatives are not considered part of the domain and need to be studied separately. For the function $f(x, y) = |xy|$ for example, we would have to look at the points on the coordinate axes separately.

1 Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x, y) = \langle 4(x^3 - y), 4(y^3 - x)\rangle$ with critical points $(0, 0), (1, 1), (-1, -1)$.

2 $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = \langle 2x \cos(x^2 + y), \cos(x^2 + y) + 1\rangle$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\dots (0, -\pi), (0, \pi), (0, 3\pi), \dots$.

3 The graph of $f(x, y) = (x^2 + y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = \langle 2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2)\rangle e^{-x^2-y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$. This function has infinitely many critical points.

4 The function $f(x, y) = y^2/2 - g \cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g \sin(x)) = \langle(0, 0\rangle$ for $(x, y) = \dots, (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \dots$. These points are equilibrium points, angles for which the pendulum is at rest.

5 The function $f(x, y) = a \log(y) - by + c \log(x) - dx$ is left invariant by the flow of the **Volterra-Lodka** differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point. It is a place, where the differential equation has stationary points.

6 The function $f(x, y) = |x| + |y|$ is smooth on the first quadrant. It does not have critical points there. The function has a minimum at $(0, 0)$ but it is not in the domain, where f and ∇f are defined.

In one dimension, we needed $f'(x) = 0, f''(x) > 0$ to have a local minimum, $f'(x) = 0, f''(x) < 0$ for a local maximum. If $f'(x) = 0, f''(x) = 0$, then the critical point was undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat inflection point like for $f(x) = x^3$.

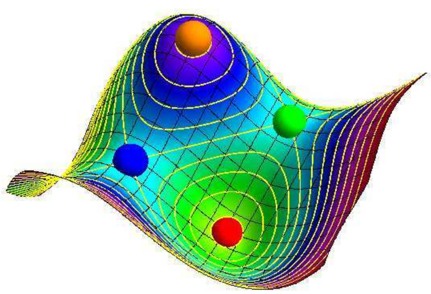
Let now $f(x,y)$ be a function of two variables with a critical point (a,b). Define $D = f_{xx}f_{yy} - f_{xy}^2$. It is called the **discriminant** of the critical point.

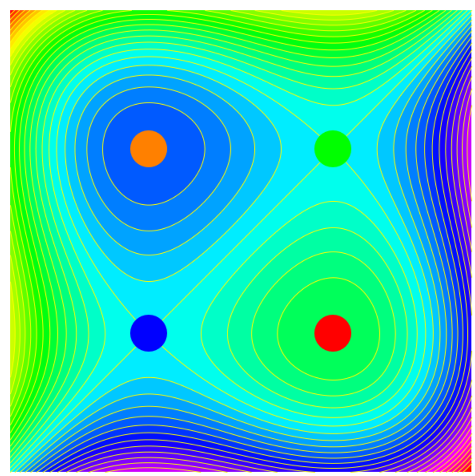
Remark: The discriminant can be remembered better if it is seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

Second derivative test. Assume (a,b) is a critical point for $f(x,y)$.
If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum.
If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum.
If $D < 0$ then (a,b) is a saddle point.

In the case $D = 0$, we need higher derivatives to determine the nature of the critical point.

7 The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x,y) = \langle x^2 - 1, -y^2 + 1\rangle$. There are 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1,1)$ we have $D = -4$ and so a saddle point,
For $(-1,1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1,-1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1,-1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





To determine the maximum or minimum of $f(x,y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see next time how to get extrema on the boundary.

8 Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3,0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3,0)$, $f(4/3,-1)$ shows that $(4/3,0)$ is the global maximum.

As in one dimensions, knowing the critical points helps to understand the function. Critical points are also physically relevant. Examples are configurations with lowest energy. Many physical laws are based on the principle that the equations are critical points. Newton equations in Classical mechanics are an example: a particle of mass m moving in a field V along a path $\gamma : t \mapsto \vec{r}(t)$ extremizes the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$ among all possible paths. Critical points γ satisfy the Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$.

Why is the second derivative test true? Assume $f(x, y)$ has the critical point $(0, 0)$ and is a quadratic function satisfying $f(0, 0) = 0$. Then

$$ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$$

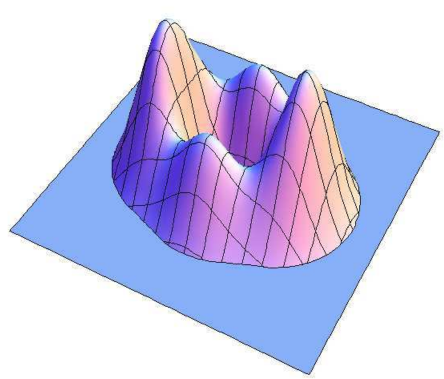
with $A = (x + \frac{b}{a}y), B = b^2/a^2$ and discriminant D. You see that if $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x, y) \neq (0, 0)$. The point $(0, 0)$ is a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x, y) \neq (0, 0)$ and the point (x, y) is a local maximum. If $D < 0$, then the function can take both negative and positive values. A general smooth function can be approximated by a quadratic function near $(0, 0)$.

Sometimes, we want to find the overall maximum and not only the local ones.

A point (a, b) in the plane is called a **global maximum** of $f(x, y)$ if $f(x, y) \leq f(a, b)$ for all (x, y). For example, the point $(0, 0)$ is a global maximum of the function $f(x, y) = 1 - x^2 - y^2$. Similarly, we call (a, b) a **global minimum**, if $f(x, y) \geq f(a, b)$ for all (x, y).

9 Does the function $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ have a global maximum or a global minimum? If yes, find them. **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x, 0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

10 Let $f(x, y)$ be the height function of an island for which only finitely many critical points exist. Assume all of them have nonzero discriminant D. Label each critical point with a $+1$ if it is a maximum or minimum, and with -1 if it is a saddle point. If you sum up all these integers you will get 1, independent of the function. This theorem of Poincare-Hopf is an example of an "index theorem", a prototype for important theorems in physics and mathematics. The indices ± 1 add up to one, whatever island you consider, whether it has only one mountain peak, or two mountain peaks and a mountain pass. Place your hand into water so that it forms an island, then and count peaks = knuckles) and mountain passes.



The following remarks can be skipped without problem:
1) if you have seen some linear algebra, you see that the discriminant D is a determinant $\det(H)$

of the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. Besides the determinant, also the trace $f_{xx} + f_{yy}$ is independent of the coordinate system. The determinant is the product $\lambda_1\lambda_2$ of the eigenvalues of H and the trace is the sum of the eigenvalues. If the determinant D is positive, then λ_1, λ_2 have the same sign and this is also the sign of the trace. If the trace is positive then both eigenvalues are positive. This means that in in the eigendirections, the graph is concave up. We have a minimum. On the other hand, if the determinant D is negative, then λ_1, λ_2 have different signs and the function is concave up in one eigendirection and concave down in an other. In any case, if D is not zero, we have an orthonormal eigenbasis of the symmetric matrix A. In that basis, the matrix H is diagonal.
2) The discriminant D can be considered also at points where we have no critical point. The number $K = D/(1 + |\nabla f|^2)^2$ is called the **Gaussian curvature** of the surface. It is remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. This is the famous Theorema Egregia (=great theorem) of Gauss. Note that at a critical point $\nabla f(x) = \vec{0}$, the discriminant agrees with the curvature $D = K$ at that point Since we mentioned the "island" already: here is an other one which follows from the Gauss-Bonnet theorem: assume you measure the curvature K at each point of an island and assume there is a nice beach all around so that the land disappears flat into the water. In that case the average curvature over the entire island is zero.
3) You might wonder what happens in higher dimensions. The second derivative test needs then more linear algebra. In three dimensions for example, one can form the second derivative matrix H and look at all the eigenvalues of H. If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. If the eigenvalues have different signs, we have a saddle point situation where in some directions the function increases and other directions the function decreases.

Homework

1 Find all the extrema of the function $f(x, y) = 4y^4 - 4x^3 - 8y^2 + 12x$ and determine whether they are maxima, minima or saddle points.

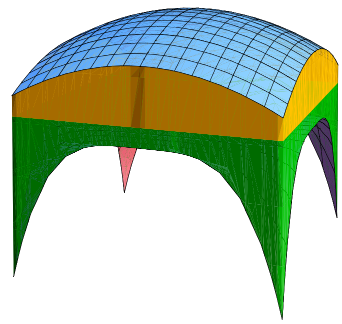
2 Where on the parametrized surface $\vec{r}(u, v) = \langle u^2, v^3, uv \rangle$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal? To find the minimum, look where the function $f(u, v) = T(\vec{r}(u, v))$ has an extremum. Find all local maxima, local minima or saddle points of f.

Remark. After you have found the function $f(u, v)$, you could replace the variables u, v again with x, y if you like and look at a function $f(x, y)$.

3 Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$.

4 Find all extrema of the function $f(x,y) = x^3+y^3-3x-12y+20$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

5 The thickness of the region enclosed by the two graphs $f_1(x,y) = 11+x^3-2x^2-2y^2$ and $f_2(x,y) = -x^4-y^4+x^3-2$ is denoted by $f(x,y) = f_1(x,y)-f_2(x,y)$. Classify all critical points of f and find the global minimal thickness.



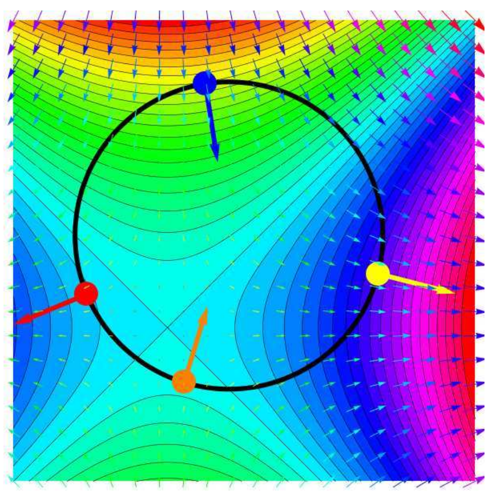
Lecture 14: Lagrange

We look now for maxima and minima of a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel because otherwise, we can move along the curve g and increase the value of f. The directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f or if there is no tangent vector.

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$. for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Lagrange theorem: A maximum or minim of $f(x, y)$ on the curve $g(x, y) = c$ is either a solution of the Lagrange equations or is a critical point of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$.



1 Minimize $f(x, y) = x^2 + 2y^2$ under the constraint $g(x, y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

2 Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = \langle 2x, 2y \rangle, \nabla g = \langle 6x^5, 6y \rangle$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1 - x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1, 0), (-1, 0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

3 Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r,h) = h\pi r^2 = 1$. The surface area is $A(r,h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. Calculate $\nabla f(x,y) = (2y, 2x + 4\pi y), \nabla g(x,y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of CO_2 dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

4 On the curve $g(x,y) = x^2 - y^3$ the function $f(x,y) = x$ obviously has a minimum $(0,0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x,y) = 0$.

Remarks.
1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
2) With $g(x,y) = 0$, the Lagrange equations can also be written as $\nabla F(x,y,\lambda) = 0$ where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$.
3) Either of the two properties equated in the Lagrange theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions. The proof is the same:

Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.

5 Find the extrema of $f(x,y,z) = z$ on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x,y,z) = (0,0,1)$, $\nabla g(x,y,z) = (2x, 2y, 2z)$ and solve $(0,0,1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0,0,-1)$ the maximum the north pole $(0,0,1)$.

6 A dice shows k eyes with probability p_k with k in $\Omega = \{1,2,3,4,5,6\}$. A probability distribution is a nonnegative function p on Ω which sums up to 1. It can be written as a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1+p_2+p_3+p_4+p_5+p_6 = 1$. The **entropy** of the probability vector $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^6 p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + ... + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy. Maximal entropy means **least information content**. An unfair dice allows a cheating gambler or casino to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

7 Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature tries to minimize the **free energy** $f(p_1, \dots, p_n) =$

$-\sum_i[p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1-\lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The Gibbs solution is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$.

Remarks:

1) Can we avoid Lagrange? Sometimes. It is often done in single variable calculus. To extremize xy under the constraint $2x + 2y = 4$ for example, we solve for y in the second equation and extremize the single variable problem $f(x, y(x))$. This needs to be done carefully and the boundaries must be considered. To extremize $f(x, y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution. To extremize $f(x, y) = x^2 + y^2$ with constraint $g(x, y) = x^4 + 3y^2 - 1 = 0$ for example, we solve $y^2 = (1 - x^4)/3$ and minimize $h(x) = f(x, y(x)) = x^2 + (1 - x^4)/3$. $h'(x) = 0$ gives $x = 0$. The find the maximum $(\pm 1, 0)$, we had to maximize $h(x)$ on $[-1, 1]$, which occurs at ± 1.

To extremize $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = p(x) + p(y) = 1$, where p is a complicated function in x which satisfies $p(0) = 0, p'(1) = 2$,the Lagrange equations $2x = \lambda p'(x), 2y = \lambda p'(y), p(x) + p(y) = 1$ can be solved with $x = 0, y = 1, \lambda = 1$. We can not solve $g(x, y) = 1$ however for y in an explicit way.

2) How do we determine whether a solution of the Lagrange equations is a maximum or minimum? Instead of using a second derivative test, we make a list of critical points and pick the maximum and minimum. A second derivative test can be designed using second directional derivative in the direction of the tangent.

3) The Lagrange method also works with more constraints. With two constraints the constraint $g = c, h = d$ defines a curve in space. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f. Here is a formulation in three dimensions.

Extrema of $f(x, y, z)$ under the constraint $g(x, y, z) = c, h(x, y, z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x, y, z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

Homework

1 Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

2 Find the extrema of the same function

$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

as in problem 4.1.3 but now on the entire disc $\{x^2+y^2 \leq 4\}$ of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

3 Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \; .$$

Is there a global maximum or a global minimum for $f(x,y)$?

4 A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \; .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

5 Minimize the material cost of an office tray

$$f(x,y) = xy + x + 2y$$

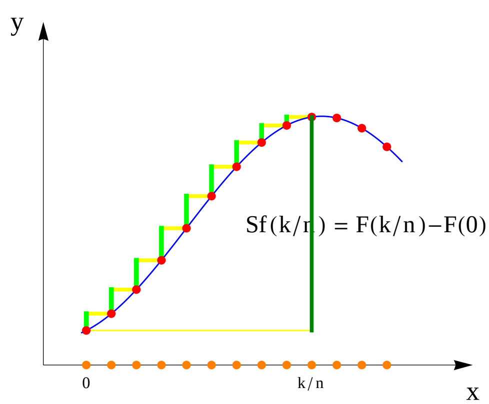
of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is constant and equal to 4.

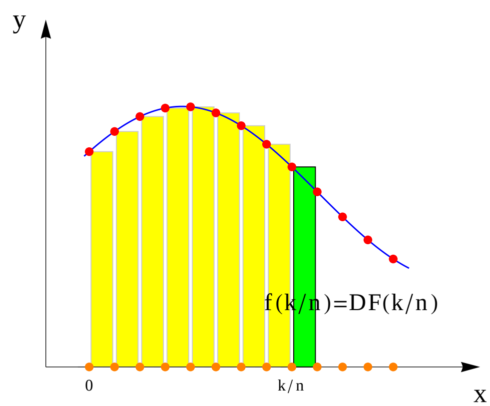
Lecture 15: Double integrals

We start with a review of single variable calculus: if $f(x)$ is a differentiable function, then the **Riemann integral** $\int_a^b f(x)\ dx$ is defined as the limit of the **Riemann sums** $S_n f(x) = \frac{1}{n}\sum_{k/n\in[a,b]} f(k/n)$ for $n \to \infty$. The **derivative** of f is the limit of **difference quotients** $D_n f(x) = n[f(x+1/n) - f(x)]$ as $n \to \infty$. The integral $\int_a^b f(x)\ dx$ is the **signed area** under the graph of f and above the x-axes, where "signed" indicates that parts below have a negative sign. The function $F(x) = \int_0^x f(y)\ dy$ is called an **anti-derivative** of f. It is determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x) = F(x) - F(0) ,$$

and allows to compute integrals by inverting differentiation. **Differentiation rules** become **integration rules**: the product rule leads to integration by parts, the chain rule becomes partial integration. For a 20×20 second version, see www.math.harvard.edu/˜ knill/pedagogy/pechakucha, For a 140 character version, see https://twitter.com/oliverknill/status/320289197653106688.





If $f(x,y)$ is differentiable on a region R, the integral $\int_R f(x,y)\ dxdy$ is defined as the limit of the Riemann sum

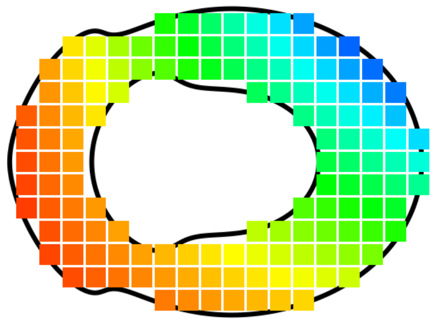
$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

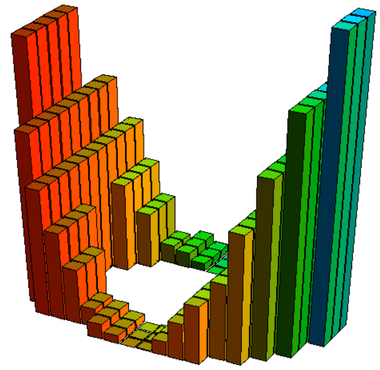
when $n \to \infty$. We write also $\int_R f(x,y)\ dA$ and think of dA as an area element.

1 If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\iint_R f(x,y)\, dA$ divided by the area of R is the **average** value of f on R.





4 One can interpret $\int\int_R f(x,y)\, dydx$ as the **signed volume** of the solid below the graph of f and above R in the $x-y$ plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle if the function f is continuous: $\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$.

Proof. We have for every n the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

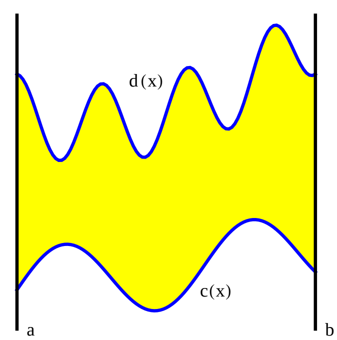
which holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.
Fubini's theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

A **type I region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \}\ .$$

An integral over such a region is called a **type I integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx\ .$$

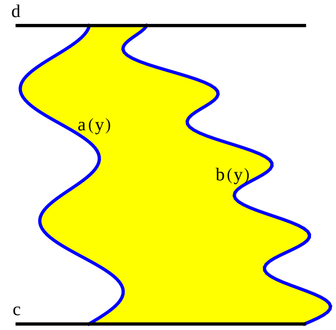


A **type II region** is of the form

$$R = \{(x,y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\int\int_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



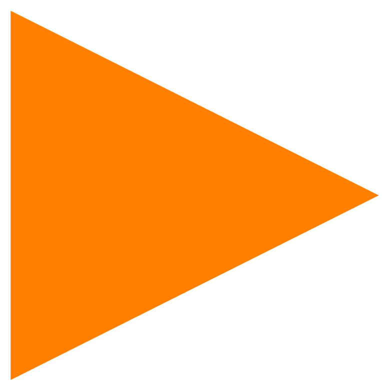
5 Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

We have now an integral, which we can solve by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} .$$

6 Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x + 1$ and $y = 1 - x$. The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y - 1$ and $x = 1 - y$



$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4} - \frac{1}{3}) = \frac{1}{10} .$$

7 Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

$$\int\int_R e^{-x^2}\ dxdy\ ?$$

The type II integral $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.

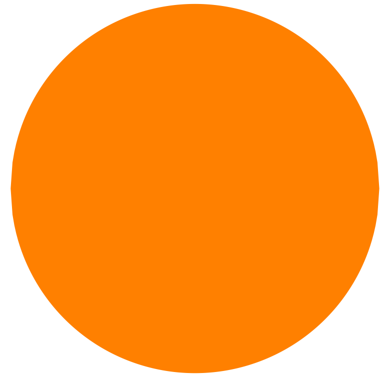
The type I integral $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$

8 The area of a disc of radius R is

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \; dydx = \int_{-R}^{R} 2\sqrt{R^2 - x^2} \; dx \; .$$

Substitute $x = R\sin(u), dx = R\cos(u)$, to get $\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)} R\cos(u) \; du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u) \; du$. Using a double angle formula. this gives $R^2 \int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2} \; du = R^2\pi$.



Remark: The Riemann integral just defined works well for continuous functions. In other branches of mathematics like probability theory, a better integral is needed. The **Lebesgue integral** fits the bill. Its definition is close to the Riemann integral which we have given as the limit $n^{-2} \int_{(x_k,y_l)\in R} f(x_k, y_l)$ where $x_k = k/n, y_l = l/n$. The Lebesgue integral replaces the regularly spaced (x_k, y_l) grid with random points x_k, y_l and uses the same formula. The following Mathematica code computes the integral $\int_0^1 \int_0^1 x^2y$ using this **Monte Carlo definition** of the Lebesgue integral.

```
M=10000; R:=Random[]; f[x_,y_]:=x^2 y; Sum[f[R,R],{M}]/M
```

It is as elegant than the numerical Riemann sum computation

```
M=100; f[x_,y_]:=x^2 y; Sum[f[k/M,l/M],{k,M},{l,M}]/M^2
```

but the Lebesgue integral is usually closer to the actual answer 1/6 than the Riemann integral. Note that for all continuous functions, the Lebesgue integral gives the same results than the Riemann integral. It does not change calculus. But it is useful for example to compute nasty integrals like the area of the Mandelbrot set.

Homework

1 Find the double integral $\int_1^4 \int_0^2 (3x - \sqrt{y}) \; dxdy$.

2 Find the area of the region

$$R = \{(x, y) \mid 0 \le x \le 2\pi, \sin(x) - 1 \le y \le \cos(x) + 2\}$$

and use it to compute the average value $\int\int_R f(x, y) \; dxdy/\text{area}(R)$ of $f(x, y) = y$ over that region.

3 Find the volume of the solid lying under the paraboloid $z = x^2+y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3] = \{(x, y) \mid -2 \le x \le 2, -3 \le y \le 3 \}$.

4 Calculate the iterated integral $\int_0^1 \int_x^{2-x} (x^2 - y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5 Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}}\ dydx\ .$$

Lecture 16: Surface area

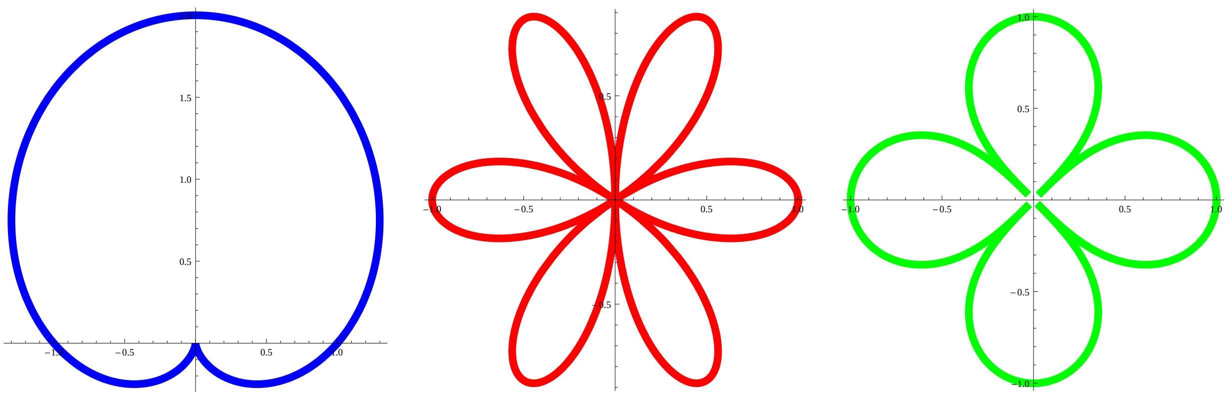
A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

In Cartesian coordinates the parametrization of the boundary of a polar region is $\vec{r}(t) = \langle r(t)\cos(\theta(t), r(t)\sin$ a **polar graph** like the spiral with $\theta(t) = t$.

1 The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

2 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

3 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign.



To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y)\ dxdy = \iint_R f(r\cos(\theta), r\sin(\theta)r\ drd\theta$$

4 Integrate

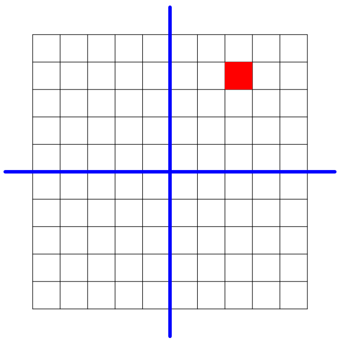
$$f(x,y) = x^2 + y^2 + xy ,$$

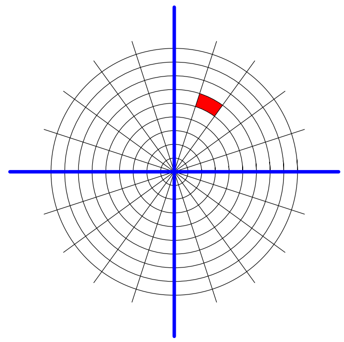
over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

5 We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\ \}$ using substitution. It is more elegant to do this integral in polar coordinates:

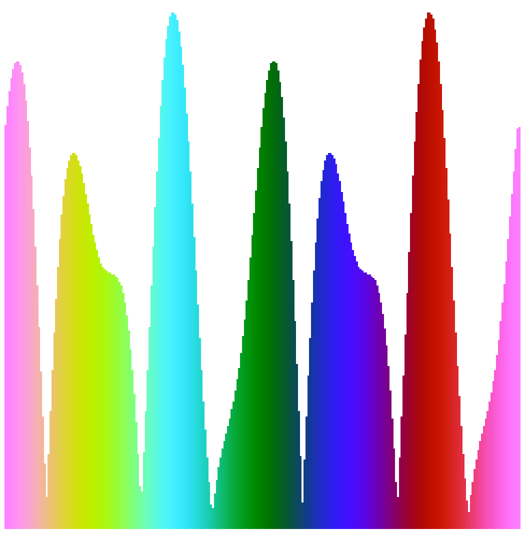
$$\int_0^{2\pi} \int_0^1 r\ drd\theta = 2\pi r^2/2|_0^1 = \pi\ .$$

Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\ d\theta dr$. If you have seen some linear algebra, note that the Jacobean matrix dT has the determinant r.

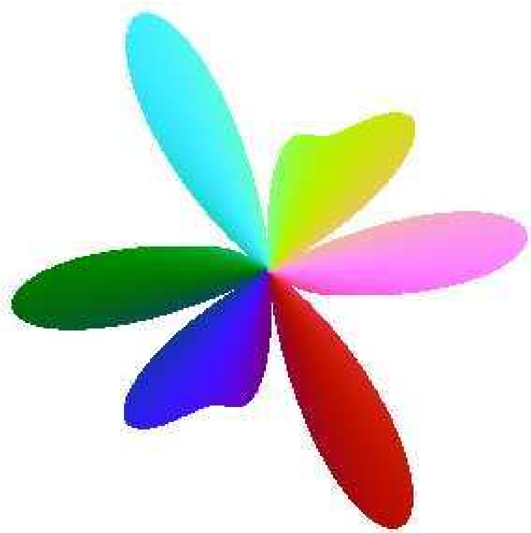




We can now integrate over type I or type II regions in the (θ,r) plane. like **flowers**: $\{(\theta,r)\ |0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



A polar region shown in polar coordinates. It is a type I region.



The same region in the xy coordinate system is not type I or II.

6 Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi}\int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2\ .$$

7 Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2+y^2 < 4, y > 0\ \}$.

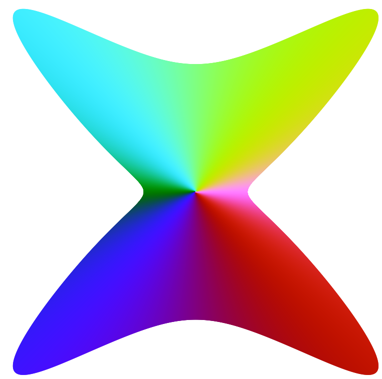
$$\int_1^2\int_0^{\pi} r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\ d\theta dr = \frac{(2^4-1^4)}{4}\int_0^{\pi}\sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms x^2+y^2 try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

8 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \, .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r}+\vec{r}_u]$ and $[\vec{r}, \vec{r}+\vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv$.

9 The parametrized surface $\vec{r}(u,v) = \langle 2u, 3v, 0 \rangle$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 6 \rangle$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

[1] "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

10 The map $\vec{r}(u,v) = \langle L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)\rangle$ maps the rectangle $G = [0,2\pi]\times[0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1\ dS = \int_0^{2\pi}\int_0^{\pi} L^2\sin(v)\ dvdu = 4\pi L^2$

11 For graphs $(u,v)\mapsto \langle u,v,f(u,v)\rangle$, we have $\vec{r}_u = (1,0,f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u,-f_v,1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1+f_u^2+f_v^2}\ dudv$.

12 Lets take a surface of revolution $\vec{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u)\rangle$ on $R = [0,2\pi]\times[a,b]$. We have $\vec{r}_u = (0,-f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v),\cos(u),\sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v|\ dudv = 2\pi\int_a^b |f(v)|\sqrt{1+f'(v)^2}\ dv$.

Homework

1 Integrate $f(x,y) = 4x^2 + 8y^2$ over the unit disc $\{x^2+y^2 \le 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

2 Find $\int\int_R (x^2+y^2)^{40}\ dA$, where R is the part of the unit disc $\{x^2+y^2 \le 1\}$ for which $y > x$.

3 What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0,2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0,2\pi]$ and third by the positive x-axis?

4 Find the average value of $f(x,y) = 2(x^2+y^2)$ on the annular region $R: 1 \le |(x,y)| \le 2$. The average is $\int_R f\ dxdy / \int_R 1\ dxdy$.

5 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \le 9$.

Homework 4 Coversheet **Maths 21a, Summer 2013**

Name:

The week 4 homework set is due July 23, 2013. Even so we had an exam, start to work early on the homework. At the end, please copy your HW answers to this cover sheet and staple this first page to the homework of lectures 4.1 and 4.2:

Homework 4.1

1	
2	
3	
4	
5	

Homework 4.2

1	
2	
3	
4	
5	

Name:

The first week homework is due July 23, 2013. Start working early on the homework! At the end, please copy your HW answers to this cover sheet and staple this first page to the homework of lectures 4.3 and 4.4:

Homework 4.3

1
2
3
4
5

Homework 4.4

1
2
3
4
5