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| Name: |
|-------|

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-2, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
| 5      |  | 10  |
| 6      |  | 10  |
| 7      |  | 10  |
| 8      |  | 10  |
| 9      |  | 10  |
| Total: |  | 100 |

Problem 1) (20 points) No justifications are needed.

1) ☐ T ☒ F  $\langle 1, 1, 1 \rangle \times \langle 1, 1, 1 \rangle = 3.$

**Solution:**

It would be true for the dot product.

2) ☒ T ☐ F The unit normal vector  $\vec{N}$  is always perpendicular to the unit tangent vector  $\vec{T}$ .

**Solution:**

Yes, we have shown this in class.

3) ☐ T ☒ F The point  $(x, y, z) = (1, 1, -\sqrt{2})$  is in spherical coordinates given by  $(\rho, \phi, \theta) = (2, \pi/4, \pi/4).$

**Solution:**

Yes, this works nicely except for the  $\phi$  angle.

4) ☐ T ☒ F Since derivatives are related to smoothness of curves, "**Don't be a  $d^3/dx^3$** " translates to "**Don't be so smooth**".

**Solution:**

We have seen this in the lecture. No, the meaning is not smart.

5) ☐ T ☒ F The curvature is a vector which points to the center of a circle which is most snug to the curve.

**Solution:**

The curvature is a scalar.

6) ☒ T ☐ F The triple scalar product of the vectors  $\vec{u}, \vec{v}, \vec{w}$  is equal to  $|u||v||w| \sin(\alpha) \cos(\beta)$  where  $\alpha$  is the angle between  $u$  and  $v$  and  $\beta$  is the angle between  $w$  and the  $\vec{u} \times \vec{v}$ .

**Solution:**

Use the formula for the length of the cross product and the cos-formula for the dot product.

- 7) 

|   |   |
|---|---|
| T | F |
|---|---|

 With the dot product we can determine the length of a vector as well as the angle between two unit vectors.

**Solution:**

Yes  $\sqrt{v \cdot v}$  is the length and  $\cos(\alpha) = \vec{v} \cdot \vec{w}$  is the angle between two unit vectors.

- 8) 

|   |   |
|---|---|
| T | F |
|---|---|

 The Cauchy-Schwartz inequality assures that  $\langle 1, 2, 3 \rangle \cdot \langle 2, 3, 4 \rangle \leq \sqrt{1^2 + 2^2 + 3^2} \sqrt{2^2 + 3^2 + 4^2}$ .

**Solution:**

Yes, this is a direct application of the inequality.

- 9) 

|   |   |
|---|---|
| T | F |
|---|---|

 The surface  $x^2 + y^2 + z^2 + 2z = -3$  is a sphere.

**Solution:**

Complete the square.

- 10) 

|   |   |
|---|---|
| T | F |
|---|---|

 The arc length of  $\langle \cos(t), t \rangle$  on the parameter interval  $[0, 1]$  is the same than the arc length of  $\langle \cos(2t), 2t \rangle$  on the interval  $[0, 2]$ .

**Solution:**

It is a different path.

- 11) 

|   |   |
|---|---|
| T | F |
|---|---|

 A plane and a line always intersect in space.

**Solution:**

They can be parallel.

- 12) 

|   |   |
|---|---|
| T | F |
|---|---|

 The line  $(x - 1) = (y - 2) = (z - 3)$  hits the plane  $x + y + z = 1$  at a right angle.

**Solution:**

The line contains the vector  $\langle 1, 1, 1 \rangle$ .

- 13) ☒ T ☐ F      The planes  $2x + 6y + 4z = 1$  and  $x + 3y + 2z = 5$  are parallel.

**Solution:**

Yes they have both the same normal vector.

- 14) ☒ T ☐ F      The parametrized curve  $\langle \cos(t), \sin(t), \sin(t) \rangle$  is an ellipse, which is obtained by intersecting the plane  $y = z$  with  $x^2 + y^2 = 1$ .

**Solution:**

Indeed, both equations are satisfied.

- 15) ☐ T ☒ F      All hyperboloids and paraboloids are graphs  $z = f(x, y)$ .

**Solution:**

No, the hyperboloids are not graphs.

- 16) ☐ T ☒ F      The vector  $\langle 3/5, 1, 4/5 \rangle$  is a unit vector.

**Solution:**

Yes, its length is equal to 1.

- 17) ☐ T ☒ F      Two vectors  $\vec{v}$  and  $\vec{w}$  are parallel if  $\vec{v} \cdot \vec{w} = 0$ .

**Solution:**

No, they are orthogonal in that case.

- 18) ☐ T ☒ F      If  $\vec{u}, \vec{v}$  are two vectors, then  $\vec{u}, \vec{v}, \vec{u} + \vec{v}$  span a parallelepiped of positive volume.

- 19) ☒ T ☐ F The plane parametrized by  $\vec{r}(t, s) = t\langle 1, 0, 0 \rangle + s\langle 0, 0, 1 \rangle$  is the same than  $y = 0$ .

**Solution:**

Indeed, the vector  $\langle 0, 1, 0 \rangle$  is perpendicular and the plane goes through the origin.

- 20) ☒ T ☐ F The partial derivative of  $f(x, y) = \sin(x^2y^2)$  with respect to  $x$  is equal to  $2xy^2 \cos(x^2y^2)$ .

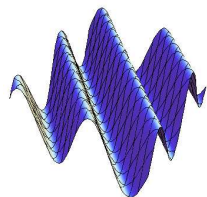
**Solution:**

Yes, we treat  $y$  as a constant and get that derivative.

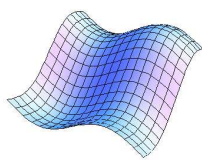
Total

Problem 2) (10 points) No justifications are needed in this problem.

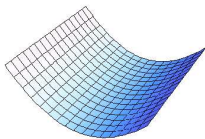
a) (2 points) Match the graphs of the functions  $f(x, y)$ . Enter O, if there is no match.



I



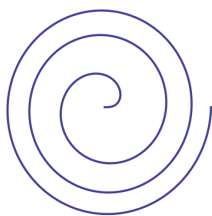
II



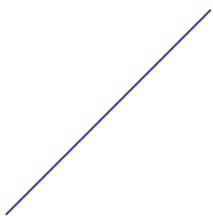
III

| Function $f(x, y) =$ | Enter O,I,II or III |
|----------------------|---------------------|
| $\sin(x + y)$        |                     |
| $x^2 - y$            |                     |
| $x^2$                |                     |
| $y^4 - 2x^2$         |                     |
| $\sin(x) + \sin(y)$  |                     |

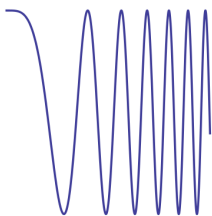
b) (3 points) Match the plane curves with their parametrizations  $\vec{r}(t)$ . Enter O, if there is no match.



I



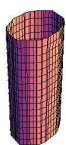
II



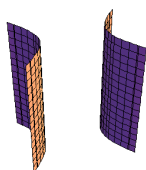
III

| Parametrization $\vec{r}(t) =$                                    | O, I,II,III |
|-------------------------------------------------------------------|-------------|
| $\vec{r}(t) = \langle \sqrt{t} \cos(t), \sqrt{t} \sin(t) \rangle$ |             |
| $\vec{r}(t) = \langle t^2, t \rangle$                             |             |
| $\vec{r}(t) = \langle t, \cos(t^2) \rangle$                       |             |
| $\vec{r}(t) = \langle 2t, 3t \rangle$                             |             |

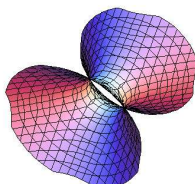
c) (2 points) Match functions  $g$  with level surface  $g(x, y, z) = 1$ . Enter O, if no match.



I



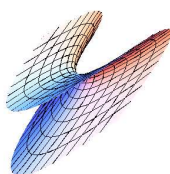
II



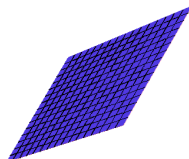
III

| Function $g(x, y, z) = 1$  | O, I,II,III |
|----------------------------|-------------|
| $g = 2x^2 + y^2 = 1$       |             |
| $g = xy = 1$               |             |
| $g = x^2 - y^2 + z^2 = 1$  |             |
| $g = -x^2 + y^2 - z^2 = 1$ |             |

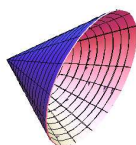
d) (3 points) Match the parametrization. Enter O, where no match.



I



II



III

| $\vec{r}(s, t)$                                             | O,I,II,III |
|-------------------------------------------------------------|------------|
| $\langle t, s, t + s \rangle$                               |            |
| $\langle t, t^2 - s^2, s \rangle$                           |            |
| $\langle s, s \cos(t), s \sin(t) \rangle$                   |            |
| $\langle \cos(t) \sin(s), \sin(t) \sin(s), \cos(s) \rangle$ |            |

**Solution:**

a) I,0,III,0,II

(the problem in the exam had the III also in II so that all cases I,0,(III,II),0,0, or I,0,III,0,0 would work.

b) I,O,III,II

c) I,II,III,0

d) II,I,III,0

|                        |
|------------------------|
| Problem 3) (10 points) |
|------------------------|

Routine problems:

- a) (2 points)  $\langle 1, 2, 3 \rangle \cdot \langle 3, 2, 1 \rangle$
- b) (2 points)  $\langle 1, 2, 3 \rangle \times \langle 3, 2, 1 \rangle$
- c) (2 points)  $\langle 1, 2, 3 \rangle \cdot \langle 3, 2, 1 \rangle \times \langle 2, 1, 3 \rangle$
- d) (2 points) The vector projection of  $\langle 1, 2, 3 \rangle$  onto the vector  $\langle 3, 2, 1 \rangle$
- e) (2 points) The cosine of the angle between  $\langle 1, 2, 3 \rangle$  and  $\langle 3, 2, 1 \rangle$ .

**Solution:**

- a) 10
- b)  $\langle -4, 8, -4 \rangle$ .
- c)  $-12$ .
- d)  $\langle 15/7, 10/7, 5/7 \rangle$ .
- e)  $5/7$ .

|                        |
|------------------------|
| Problem 4) (10 points) |
|------------------------|

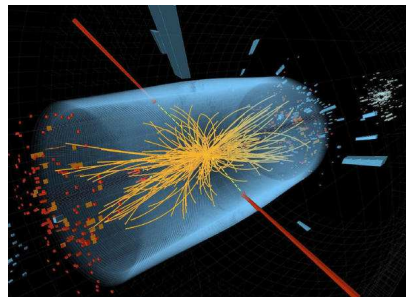
- a) (7 points) Find the distance  $d$  between the plane  $x + 2y + 2z = 1$  and the point  $P = (0, 2, 3)$ .
- b) (3 points) Find the radius of the circle obtained by intersecting the sphere of radius 5 centered at  $P$  with the plane.

**Solution:**

- a) The point  $(1, 0, 0)$  is on the plane. Take the distance formula  $d = |\vec{n} \cdot \vec{PQ}|/|\vec{n}| = \boxed{3}$ .
- b) Pythagoras for the triangle with sphere radius 5 as hypotenuse and distance  $d$  as one cathetus 3 gives the radius of the circle as the other cathetus:  $r = \sqrt{5^2 - 3^2} = \boxed{4}$ .

|                        |
|------------------------|
| Problem 5) (10 points) |
|------------------------|

a) (5 points) On July 4, the discovery of the Higgs boson was announced. A proton on the large hadron collider in Geneva is subject to a force  $\vec{r}''(t) = \langle \cos(t), \sin(t), 0 \rangle$  which is imposed on the particle by strong magnets. Assume it is at  $\vec{r}(0) = \langle 1, 1, 10 \rangle$  at time  $t = 0$  and  $\vec{r}'(0) = \langle 0, 0, 1 \rangle$ . Where is the particle at  $t = \pi$ ?



b) (5 points) Write down the arc length integral of the parametrized curve  $\vec{r}(t)$  for which  $t \in [0, 2\pi]$ .

**Solution:**

a) Integrate twice. First get

$$\vec{r}'(t) = \int_0^t \vec{r}''(s) ds + \vec{r}'(0) = \langle \sin(t), 1 - \cos(t), 0 \rangle + \langle 0, 0, 1 \rangle$$

Now integrate again

$$\vec{r}(t) = \int_0^t \vec{r}'(s) ds + \vec{r}(0) = \langle 2 - \cos(t), 1 + t - \sin(t), 10 + t \rangle.$$

Plug in the time  $t = \pi$  to get  $\vec{r}(\pi) = \langle 3, 1 + \pi, 10 + \pi \rangle$ .

b) First get the speed  $|\vec{r}'(t)| = \sqrt{\sin^2(t) + (1 - \cos(t))^2 + 1} = \sqrt{3 - 2\cos(t)}$ . The arc length integral is  $\int \sqrt{3 - 2\cos(t)} dt$ .

Problem 6) (10 points)

a) (5 points) Parametrize a line perpendicular to the plane  $x - y + z = 5$  which is at time  $t = 0$  at the point  $(0, 0, 2)$ .

b) (5 points) Intersect this line with the sphere  $x^2 + y^2 + z^2 = 36$ .

**Solution:**

a) The parametrization is  $\vec{r}(t) = \langle t, -t, 2 + t \rangle$ . This is  $x = t, y = -t, z = 2 + t$ .

b) Plug  $x, y, z$  into the sphere  $x^2 + y^2 + z^2 = 1$  and solve the corresponding quadratic equation for  $t$ . This gives  $t = 8/3$  and  $t = -4$ . We have  $\vec{r}(8/3) = \langle 8/3, -8/3, 14/3 \rangle$ .

Problem 7) (10 points)

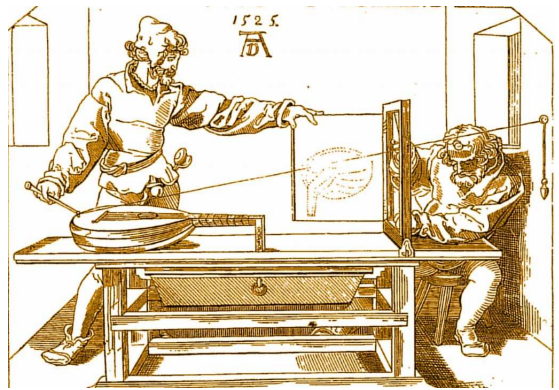


The plane  $\Sigma : x + 2y + z = 1$  is the photographic plate and  $O = (0, 1, 0)$  is the viewpoint. For a point like  $P = (3, 4, 5)$  we form the line through  $O$  and  $P$  and intersect with the plane.

a) (5 points) Parametrize the line  $L$  as  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ .

b) (5 points) Intersect the line  $L$  with the plane  $\Sigma$  to get the image point on the plane.

Picture "Perspective machine" by Albrecht Dürer, 1525



### Solution:

a)  $\vec{r}(t) = \langle 3t, 1 + 3t, 5t \rangle$ .

b) To get the time  $t$ , plug in  $x(t) = 3t, y(t) = 1 + 3t, z(t) = 5t$  into  $x + 2y + z = 1$  to solve for  $t = -1/14$ . So,  $\boxed{\vec{r}(-1/14) = \langle -3/14, 11/14, -5/14 \rangle}$ .

### Problem 8) (10 points)

Find the parametrizations  $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ . You can of course use other variables than  $u, v$ . For example, if  $y = \sin(xz)$ , then  $\vec{r}(x, z) = \langle x, \sin(xz), z \rangle$ .

a) (2 points) The hyperbolic paraboloid  $z = x^2 - y^2$ .

b) (2 points) The one sheeted paraboloid  $x^2 + y^2 - z^2 = 1$ .

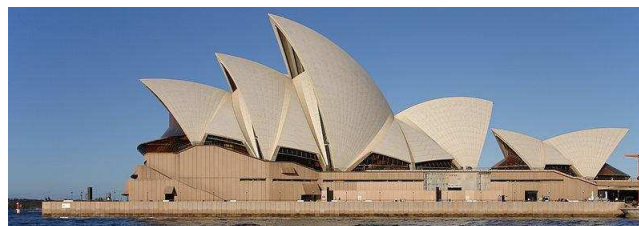
c) (2 points) The ellipsoid  $(x - 1)^2 + y^2 + \frac{z^2}{4} = 1$

d) (2 points) The cone  $x^2 + z^2 = y^2$ .

e) (2 points) A roof the Sidney opera house in Sidney is parametrized as

$$\vec{r}(t, s) = \langle \cos(t) \sin(s), \cos(s), \sin(t) \sin(s) \rangle .$$

What surface is that?



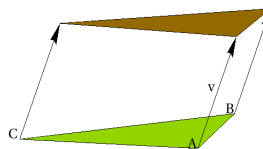
**Solution:**

- a) This is a graph  $\vec{r}(x, y) = \langle x, y, x^2 - y^2 \rangle$ .  
 b) This is a surface of revolution  $\vec{r}(\theta, z) = \langle \sqrt{1+z^2} \cos(\theta), \sqrt{1+z^2} \sin(\theta), z \rangle$ .  
 c) Deform and translate the sphere  $\vec{r}(\theta, \phi) = \langle 1 + \cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), 2 \cos(\phi) \rangle$ . d) Also this is a surface of revolution  $\vec{r}(\theta, y) = \langle y \cos(\theta), y, y \sin(\theta) \rangle$ .  
 e) This is a sphere. We just have the  $y$  and  $z$  coordinates switched but this does not change the fact that  $x^2 + y^2 + z^2 = 1$ .

## Problem 9) (10 points)

- a) (5 points) What is the area of the triangle through the points  $A = (1, 1, 1)$  and  $B = (0, 1, 0)$  and  $C = (1, 2, 4)$ .

- b) (5 points) Find the volume of the prism which has the triangle  $T$  as base as well as a by  $\vec{v} = \langle 0, 1, 1 \rangle$  translated triangle as roof.

**Solution:**

- a) It is half the length of

$$\vec{AB} \times \vec{AC} = \langle -1, 0, -1 \rangle \times \langle 0, 1, 3 \rangle = \langle 1, 3, -1 \rangle$$

which has length  $\sqrt{11}$  so that the answer is  $\boxed{\sqrt{11}/2}$ .

- b) The volume is half of the volume of the parallel piped and therefore half of the absolute value of the triple scalar product

$$\vec{v} \cdot (\vec{AB} \times \vec{AC}) = \langle 0, 1, 1 \rangle \cdot \langle 1, 3, -1 \rangle = 2$$

so that the answer is  $\boxed{1}$ .