7/11/2013 FIRST HOURLY Practice III **Maths 21a, O.Knill, Summer 2013**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T **F** The length of the vector $\langle 1,1,1 \rangle$ is equal to 3.

Solution:
The length is the square root of 3.

2) **T** F Any two distinct points A, B in space determine a unique line which contains these two points.

Solution:
The two points allow to give a parametrization $A + t\vec{AB}$

3) T **F** For any two non-intersecting lines L, K, there is exactly one point P which has equal distance to both lines.

Solution:
There are many points which have equal distance. Points with this property form a plane as described in a main problem below.

4) **T** F The graph of $f(x,y)$ is a surface in space which is equal to the level surface $g(x,y,z) = 0$ of $g(x,y,z) = f(x,y) - z$.

Solution:
It is the level surface to $c = 0$.

5) T **F** The graph of the function $f(x,y) = x^2 - y^2$ is called an elliptic paraboloid.

Solution:
Nope, it is a hyperbolic paraboloid. An elliptic one would have a plus sign $f(x,y) = x^2 + y^2$.

6) T **F** The equation $\rho\cos(\theta) = 1$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, the equation $\rho\cos(\phi)$ is $z = 1$ but $\rho\cos(\theta) = 1$ is not a plane.

7) | T | F | The vector $\langle 1, 2, 3\rangle$ is perpendicular to the plane $x + 2y + 3z = 4$.

Solution:
The vector $\langle 1, 2, 3\rangle$ is indeed a normal vector.

8) | T | F | The cross product between the vectors $\langle 1, 2, 3\rangle$ and $\langle 1, 1, 1\rangle$ is 6.

Solution:
It is the dot product which is 6. The cross product is a vector.

9) | T | F | The two parametrized curves $\vec{r}(t) = \langle t, t^2, t^6\rangle, 0 \le t \le 1$ and $\vec{R}(t) = \langle t^2, t^4, t^6\rangle, 0 \le t \le 1$ have the same arc length.

Solution:
This is a change of parametrization but the powers do not match.

10) | T | F | The point $(1, 0, 1)$ has the spherical coordinates $(\rho, \theta, \phi) = (\sqrt{2}, 0, \pi/4)$.

Solution:
Apply the transformation formulas. We have $\theta = -\pi/4$.

11) | T | F | The distance between two parallel planes is the distance of any point on one plane to the other plane.

Solution:
Note that this is only true for parallel lines.

12) | T | F | $\text{Proj}_{\vec{w}}(\vec{v} \times \vec{w}) = \vec{0}$ holds for all nonzero vectors $\vec{v}, \vec{w}$.

Solution:
The vector $(\vec{v} \times \vec{w})$ is projected onto a vector perpendicular to it.

13) | T (marked) | F |

The vector projection of $\langle 2, 3, 4 \rangle$ onto $\langle 1, 0, 0 \rangle$ is $\langle 2, 0, 0 \rangle$.

Solution:
Apply the formula. Because the vector on which we project has length 1, the result is the dot product times this vector.

14) | T | F (marked) |

The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ between three vectors $\vec{u}, \vec{v}, \vec{w}$ is zero if and only if two or more of the 3 vectors are parallel.

Solution:
They can be nonparallel but in the same plane.

15) | T (marked) | F |

There are two vectors $\vec{v}$ and $\vec{w}$ so that the dot product $\vec{v} \cdot \vec{w}$ is equal to the length of the cross product $|\vec{v} \times \vec{w}|$.

Solution:
Take two vectors which make an angle of 45 degrees. Then $\sin(\theta) = \cos(\theta)$ and the length of the cross product is equal to the dot product. Examples are $\vec{v} = \langle 1, 0, 0 \rangle, \vec{w} = \langle 1, 1, 0 \rangle$.

16) | T | F (marked) |

Two cylinders of radius 1 whose axes are lines L, K have distance $d(L, K) = 3$ have distance 2.

Solution:
The distance is $1 = 3 - 1 - 1$. We have to subtract the two radii of the cylinders.

17) | T | F (marked) |

There are two unit vectors $\vec{v}$ and $\vec{w}$ that are both parallel and perpendicular.

Solution:
The cross and dot product have to be zero simultaneously which is not possible.

18) T F Assuming the curvature to exist for all time, the curvature $\kappa(\vec{r}(t))$ is always smaller than or equal to $|\vec{r}\,''(t)|/|\vec{r}\,'|^2$.

Solution:
Use the formula and the fact that the cross product in norm is smaller or equal than the product $|r''(t)| \cdot |r'(t)|$.

19) T F The curve $\vec{r}(t) = \langle \cos(t)\sin(t), \sin(t)\sin(t), \sin(t) \rangle$ is located on a sphere.

Solution:
Check $x^2 + y^2 + z^2 = 1$ is false. With a $\cos(t)$ in the end, it would be true.

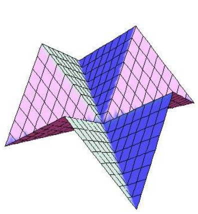
20) T F The surface $x^2 + y^2 + z^2 = 4z - 3$ is a sphere of radius 1.

Solution:
Complete the square.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.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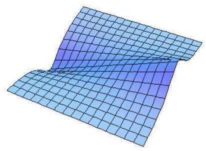
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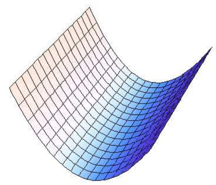
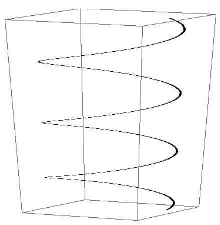

III

Function $f(x, y) =$	Enter O,I,II or III
x^2	
$\lvert\lvert x\rvert - \lvert y\rvert\rvert$	
$x^2 + y^2$	
$x/(1 + y^2)$	

b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.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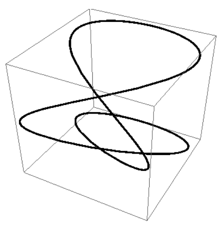
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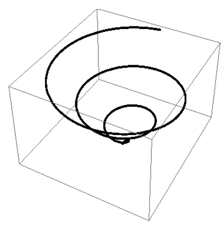
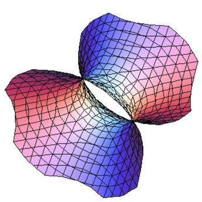

III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\vec{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$	
$\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$	
$\vec{r}(t) = \langle \sin(t), \cos(t), 0\rangle$	
$\vec{r}(t) = \langle \sin(3t), \cos(2t), \cos(t)\rangle$	

c) (2 points) Match the functions g with the level surface $g(x, y, z) = 1$. Enter O, where no match.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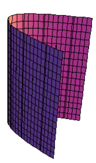
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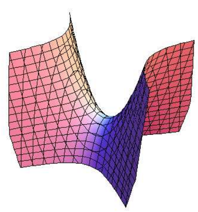
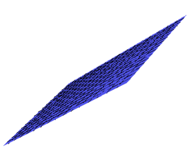

III

Function $g(x, y, z) = 1$	O, I,II,III
$g(x, y, z) = x^2 - y^2 - z = 1$	
$g(x, y, z) = x - y^2 = 1$	
$g(x, y, z) = x^2 - y^2 + z^2 = 1$	
$g(x, y, z) = x^2 + y^2 + z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.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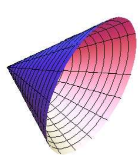
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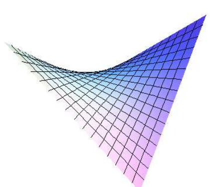

III

Function $g(x, y, z) =$	O,I,II,III
$\vec{r}(s, t) = \langle t, s, ts\rangle$	
$\vec{r}(s, t) = \langle t^2 + s^2, s, t\rangle$	
$\vec{r}(s, t) = \langle t, t\cos(s), t\sin(s)\rangle$	
$\vec{r}(s, t) = \langle t, s, t\rangle$	

Solution:
a) III,I,0,II
b) III,I,0,II
c) III,II,I,0
d) III,0,II,I

Problem 3) (10 points) No justifications are needed in this problem.

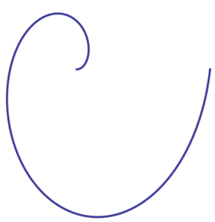
a) (2 points) Translate from polar to Cartesian coordinates or back:

Polar coordinates $(\theta, r) =$	Cartesian coordinates $(x, y) =$
$(\pi/2, 1)$	
	$(1, 1)$

b) (2 points) Translate from spherical to Cartesian coordinates or back:

Spherical coordinates $(\theta, \phi, \rho) =$	Cartesian coordinates $(x, y, z) =$
$(\pi/2, \pi/2, 1)$	
	$(1, 1, 1)$

c) (3 points) Match the curves given in polar coordinates. Enter O, if there is no match.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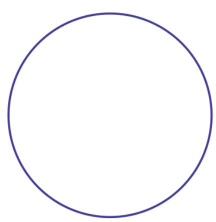
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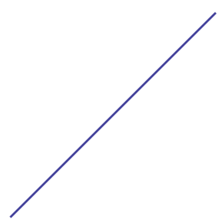
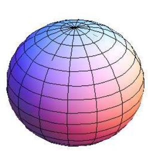

III

Surface	Enter I,II,II, O
$r = \pi/4$	
$r = -\theta$	
$\theta = \pi/4$	
$r = \theta$	

d) (3 points) Match the surfaces given in spherical coordinates. Enter O, if there is no match.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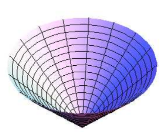
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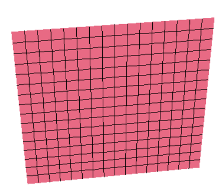
III

Surface	Enter I,II,III,O
$\rho = \pi/4$	
$\phi = \pi/4$	
$\theta = \phi$	
$\theta = \pi/4$	

Solution:

Polar coordinates $(\theta, r) =$	Cartesian coordinates $(x, y) =$
$(\pi/2, 1)$	$(0, 1)$
$(\pi/4, \sqrt{2})$	$(1, 1)$

Spherical coordinates $(\theta, \phi, \rho) =$	Cartesian coordinates $(x, y, z) =$
$(\pi/2, \pi/2, 1)$	$(0, 1, 0)$
$(\pi/4, \arcsin(\sqrt{2/3}), \sqrt{3})$	$(1, 1, 1)$

c) II,0,III,I
d) I,II,O,III

Problem 4) (10 points)

a) (2 points) Given two points $A = (1, 2, 3)$ and $B = (4, 5, 6)$. Find the midpoint M between these two points.

b) (5 points) Find the equation $ax + by + cz = d$ of the plane for which every point has equal distance to both A and B.

c) (3 points) Write down a parametrization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the line containing both A and B.

Solution:
a) The mid point of two points is $(A + B)/2 = (5/2, 7/2, 9/2)$.
b) The vector $\vec{AB} = \langle 3, 3, 3 \rangle$ is perpendicular to the plane. The plane has the form $3x + 3y + 3z = d$. By plugging in a point on the plane like $(5/2, 7/2, 9/2)$ we get the constant $d = 31.5$.
c) The vector $\vec{AB} = \langle 3, 3, 3 \rangle$ is in the line. The parametrization is

$$\vec{r}(t) = \langle 1, 2, 3 \rangle + t\langle 3, 3, 3 \rangle \ .$$

Problem 5) (10 points)

In this problem you have to find parametrizations of surfaces. The parametrization should have the form

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \ .$$

a) (2 points) Parametrize the paraboloid $z = -x^2 - y^2$.

b) (2 points) Parametrize the ellipsoid $x^2 + y^2/4 + z^2/9 = 1$.

c) (2 points) Parametrize the plane $x = y$.

d) (2 points) Parametrize the cylinder $x^2 + z^2 = 9$.

e) (2 points) Parametrize the cone $x^2 + y^2 = z^2$.

Solution:
a) This is a graph $\vec{r}(u, v) = \langle u, v, -u^2 - v^2 \rangle$.
b) This is a modification of a sphere: $\vec{r}(u, v) = \langle \cos(u)\sin(v), 2\sin(u)\sin(v), 3\sin(v) \rangle$.
c) This is a plane. A parametrization is $\vec{r}(u, v) = \langle u, u, v \rangle$.
d) This is an example of a surface of revolution with $r = 3$ so that $\vec{r}(u, v) = \langle 3\cos(u), 3\sin(u), v \rangle$.
e) Also this is a surface of revolution with $r = z$ so that $\vec{r}(u, v) = \langle v\cos(u), v\sin(u), v \rangle$.

Problem 6) (10 points)

In the soccer world championship of 2010, the English team scored a goal against the German team which the referee did not see. Assume the ball followed the line $x = y = (z - 1)/3$ and that the referee was at position $R = (3, 2, 1)$.

Remark: The incidence was a Wimbleton revanche and confirmed a word of wisdom of former player Gary Lineker who said a couple of years ago: "Soccer is a game for 22 people that run around, play the ball, and one referee who makes a slew of mistakes, and in the end, Germany always wins."

a) (4 points) Find a parametrization of the line L which the ball followed.

b) (6 points) Find the minimal distance of L from the point R.

Solution:
a) From the symmetric equations, we can read off a point $Q = (0, 0, 1)$ on the line and a vector $\vec{v} = \langle 1, 1, 3 \rangle$ inside the line. The parametrization is

$$\vec{r}(t) = \langle 0, 0, 1 \rangle + t\langle 1, 1, 3 \rangle \ .$$

b) Use the formula for the distance between a point and a vector:

$$d = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} \ .$$

Since $\vec{PQ} = \langle 3, 2, 0 \rangle$ we have

$$\vec{PQ} \times \vec{v} = \langle 3, 2, 0 \rangle \times \vec{v} = \langle 1, 1, 3 \rangle = \langle 6, -9, 1 \rangle$$

which has length $\sqrt{118}$. The distance is $\sqrt{118}/\sqrt{11}$.

Problem 7) (10 points)

Compute the following expressions:

a) (2 points) the length of the vector $\langle 1, 1, 2 \rangle$,

b) (2 points) the cross product $\langle 1, 1, 2 \rangle \times \langle 2, 2, 1 \rangle$,

c) (2 points) the dot product $\langle 1, 1, 4 \rangle \cdot \langle 2, 4, 2 \rangle$,

d) (2 points) the projection $\mathrm{Proj}_{\langle 1,1,2 \rangle}(\langle 1, 2, 3 \rangle)$,

e) (2 points) the angle between $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$.

Solution:
a) $\sqrt{6}$
b) $\langle -3, 3, 0 \rangle$
c) 14
d) $\langle 1, 1, 2 \rangle (3/2)$
e) $\cos(\alpha) = 1/2$ so that $\alpha = \pi/3$.

Problem 8) (10 points)

a) (3 points) Find the unit tangent vector $\vec{T}(t)$ of the curve

$$\vec{r}(t) = \langle t^2, 3t^3, 12t^{5/2}/5 \rangle$$

at time $t = 1$.

b) (7 points) Find the arc length of the same curve from $0 \leq t \leq 2$.

Solution:
a) $\vec{r}\,'t = \langle 2t, 9t^2, 6t^{3/2} \rangle$ and $|\vec{r}\,'(t)| = \sqrt{(4t^2 + 81t^4 + 36t^3)} = 2t + 9t^2$ so that

$$T(1) = \langle 2, 9, 6 \rangle / 11 \ .$$

b) To compute the distance, we have to integrate

$$L = \int_0^2 2t + 9t^2 \; dt = t^2 + 3t^3 |_0^2 = 4 + 24 = 28 \ .$$

Problem 9) (10 points)

In "Avatar", the "floating mountains of Pandora" contain material which is attracted by the neighboring planet. These stones would therefore float away if not tied back by tree branches or weighted with usual rocks. Pandora stones feel a force (= acceleration) $\vec{r}\,''(t) = \langle 0, 0, 2 \rangle$. Neytiri throws such a stone from the initial position $\vec{r}(0) = \langle 1, 0, 3 \rangle$ with initial velocity $\vec{r}\,'(0) = \langle 3, 4, 0 \rangle$ towards the bottom S of a floating rock $z = 100$.

a) (5 points) Find the path $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the stone.

b) (2 points) At which time t does the stone hit the rock wall S given by the equation $z = 100$?

c) (3 points) What is the angle of impact between the stone path and the normal vector $\langle 0, 0, 1 \rangle$ of S?

Solution:
a) We are given the acceleration

$$\vec{r}\,''(t) = \langle 0, 0, 2 \rangle$$

and get

$$\vec{r}\,'(t) = \langle 0, 0, 2t \rangle + \langle 3, 4, 0 \rangle \ .$$

Integrating again gives

$$\vec{r}\,(t) = \langle 0, 0, t^2 \rangle + \langle 3t, 4t, 0 \rangle + \langle 1, 0, 3 \rangle \ .$$

b) We want $z(t) = t^2 + 3 = 100$. We have $t^2 = 97$ so that $t = \sqrt{97}$.
c) The velocity at the impact is $\langle 3, 4, 2\sqrt{97} \rangle$. Therefore $\cos(\alpha) = \sqrt{388/413}$.