7/25/2013 SECOND HOURLY PRACTICE IV Maths 21a, O.Knill, Summer 2013

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** — Every critical point of a smooth function $f(x,y)$ of 2 variables is either a maximum or a minimum.

Solution:
The function $f(x,y) = x + y$ does not have a critical point.

2) **T** F — If $f(x,y) = 10$ and a region G in the plane is given, then $\iint_G f(x,y)\, dydx$ is 10 times the area of the region.

Solution:
If $f(x,y) = 1$, then we get the area.

3) T **F** — If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y)\, dxdy > 0$.

Solution:
The critical point can be surrounded by a small region only, where f is positive.

4) **T** F — If a curve $\vec{r}(t)$ cuts a level curve in a right angle and nonzero velocity at a point which is not critical, then the $d/dt f(r(t)) \neq 0$ at that point.

Solution:
This is a consequence of the chain rule: the derivative is $\nabla f(r(t)) \cdot r'(t)$. The assumption implies that $r'(t)$ is parallel to ∇f.

5) **T** F — The linearization of a function $f(x,y)$ at $(0,0)$ has a graph which is a plane $ax + by + cz = d$ tangent to the graph of $f(x,y)$.

Solution:
The linearized function is linear and approximates the function at $(0,0)$. Its level curves are planes and one of this plane is the tangent plane.

6) | T | F | The surface area $\vec{r}(u,v) = \langle u^2, v^2, u^2+v^2 \rangle$ with $0 \leq u \leq 1, 0 \leq v \leq 1$ is equal to the surface area of $\vec{r}(u,v) = \langle u^3, v^3, u^3 + v^3 \rangle$ with $0 \leq u \leq 1, 0 \leq v \leq 1$.

Solution:
Different parametrizations of the same surface have the same surface area. In this case, this is a parametrization of a plane.

7) | T | F | Assume $\vec{r}(u,v)$ is a parametrization of a surface $g(x,y,z) = d$ and $\vec{r}(1,2) = 3$, then $\nabla g(1,2,3)$ is parallel to $\vec{r}_u(1,2) \times \vec{r}_v(1,2)$.

Solution:
Both vectors are perpendicular to the surface at the point $(1,2,3)$.

8) | T | F | Any region which is both type I and type II must be a rectangle.

Solution:
By definition of type I and type II.

9) | T | F | A given function $f(x,y)$ defines two functions $g(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2$ and $h(x,y) = f_{xx}(x,y)$. Assume both g and h are positive everywhere, then every critical point of f must be a local minimum.

Solution:
This is a consequence of the second derivative theorem.

10) | T | F | If $f(x,y)$ satisfies the Laplace equation $f_{xx} = -f_{yy}$ then every critical point of f with nonzero discriminant D is a saddle point.

Solution:
Yes, then $D = -f_{xx}^2 - f_{xy}^2$ is negative.

11) | T | F | The Lagrange multiplier λ at a solution (x,y,λ) of the Lagrange equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = 0$ has the property that it is always positive or zero.

Solution:
The sign of λ changes if g is replaced by $-g$ but the critical points stay.

12) T F The partial differential equation $u_t = u_{xx}$ is called the heat equation.

Solution:
This is an important partial differential equation.

13) T F The gradient $\nabla f(x, y, z)$ of a function of three variables is a vector tangent to the surface $f(x, y, z) = 0$ if (x, y, z) is on the surface.

Solution:
It is perpendicular, not tangent.

14) T F The value of $\log(2 + x)$ with natural log can be estimated by linear approximation as $\log(2) + x/2$.

Solution:
Yes, because $\log'(x) = 1/x$ and $\log(2 + x)$ has the derivative $1/(2 + x)$.

15) T F The tangent line to the curve $f(x, y) = x^3 + y^3 = 9$ at $(2, 1)$ can be parametrized as $\vec{r}(t) = \langle 2, 1\rangle + t\langle 8, 3\rangle$ since $\langle 8, 3\rangle$ is the gradient at $(2, 1)$.

Solution:
This was the parametrization of a line perpendicular to the curve.

16) T F Any function $f(x, y)$ which has a local maximum also has a global maximum.

Solution:
There are functions which only have local maxima already in one dimensions. An example is $f(x) = x^4 - x^2$. Now take $f(x, y) = (x^2 + y^2)^4 + x^2 + y^2$.

17) | T | F | The directional derivative of a function $f(x,y)$ in the direction of the tangent vector to the level curve is zero.

Solution:
Yes, because it is then perpendicular to the curve.

18) | T | F | The directional derivative of a function in the direction $\nabla f/|\nabla f|$ of the gradient is always nonnegative at a point which is not a critical point.

Solution:
Indeed, an important fact

19) | T | F | The chain rule tells $d/dt f(t^4,t^3)|_{t=1}$ is equal to the dot product of the gradient of f at $(1,1)$ and the velocity vector $(4,3)$ at $(1,1)$.

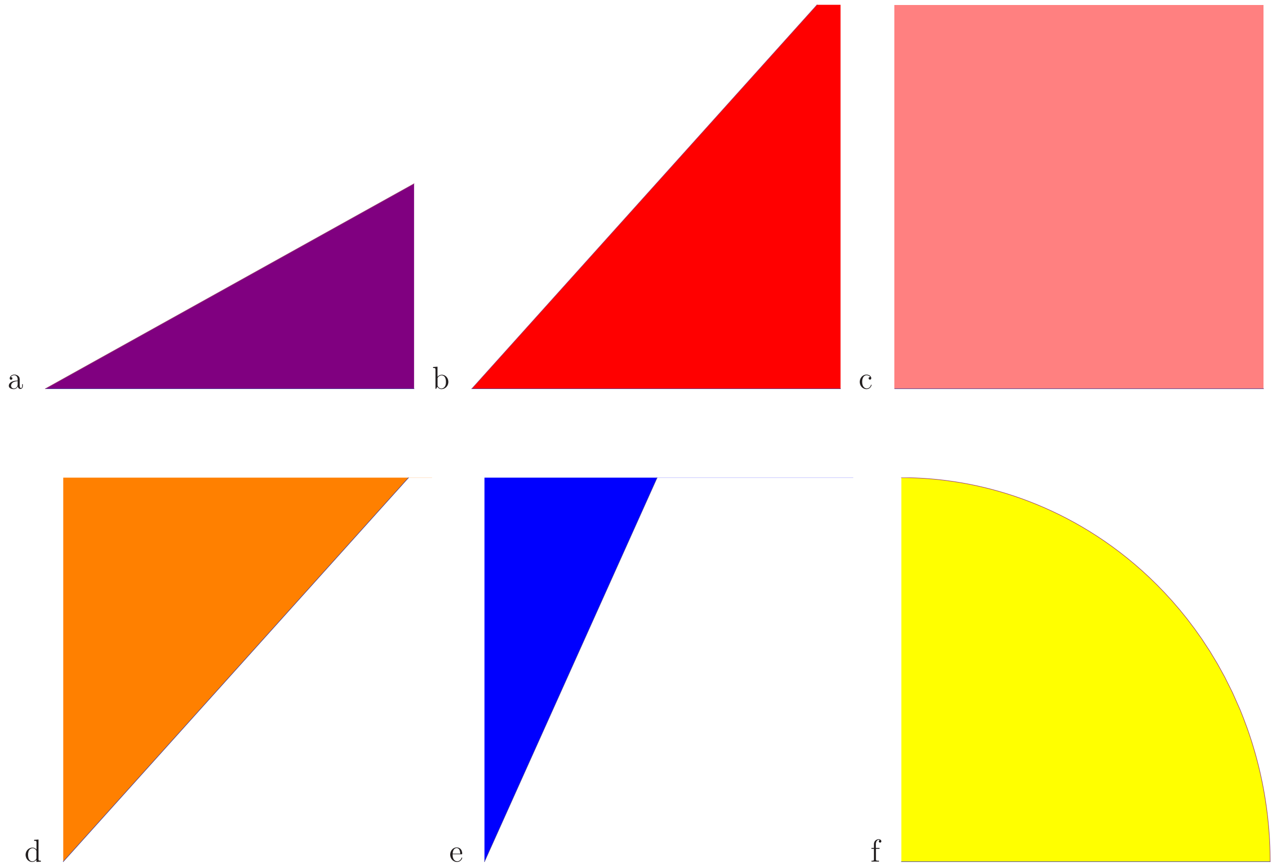
Solution:
Thats what the chain rule tells.

20) | T | F | Fubini's theorem assures that $\int_0^1 \int_x^1 f(x,y)\; dxdy = \int_0^1 \int_y^1 f(x,y)\; dydx$.

Solution:
No this is a wrong identity.

Problem 2) (10 points) No justifications are needed

Match the regions with the double integrals. Only 5 of the 6 choices match.



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta)r\ d\theta dr$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(x,y)\ dxdy$
	$\int_0^{\pi/2}\int_0^{x} f(x,y)\ dydx$
	$\int_0^{\pi/2}\int_0^{y} f(x,y)\ dxdy$
	$\int_0^{\pi/2}\int_0^{x/2} f(x,y)\ dydx$

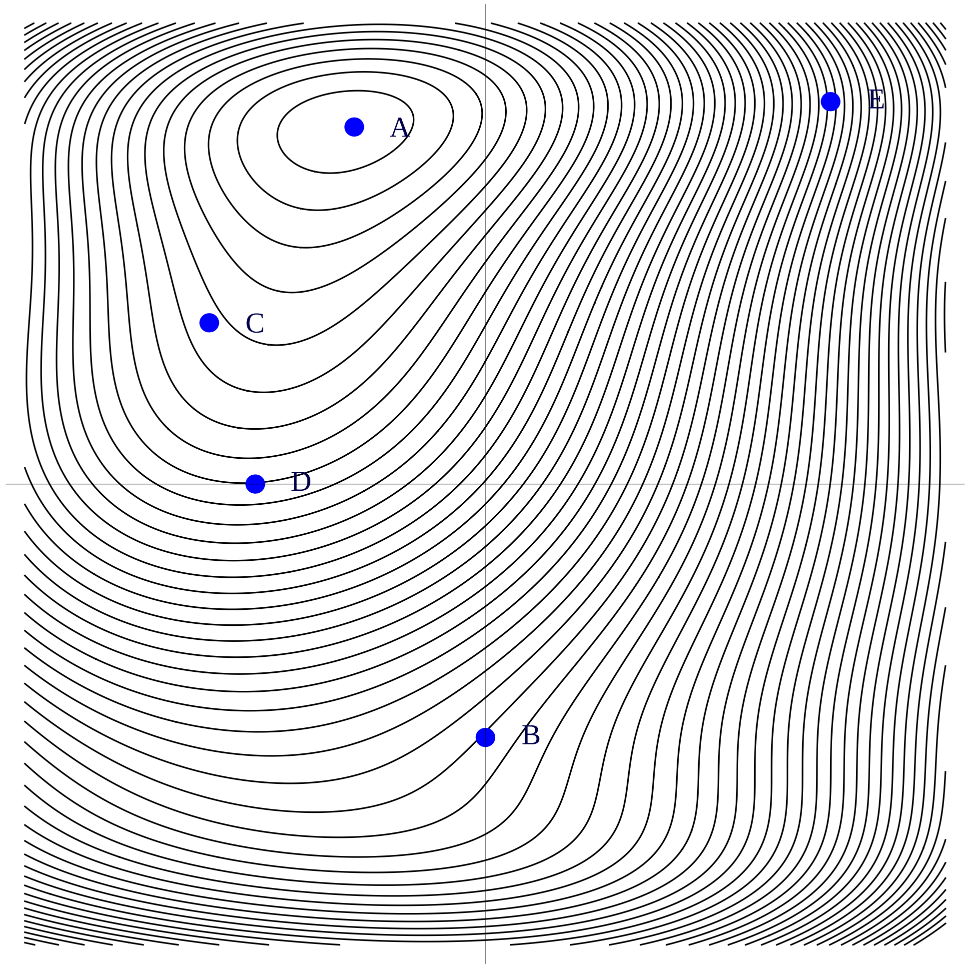
Solution:

Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
f	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta) r \, d\theta dr$
c	$\int_0^{\pi/2}\int_0^{\pi/2} f(x,y) \, dxdy$
b	$\int_0^{\pi/2}\int_0^{x} f(x,y) \, dydx$
d	$\int_0^{\pi/2}\int_0^{y} f(x,y) \, dxdy$
a	$\int_0^{\pi/2}\int_0^{x/2} f(x,y) \, dydx$

Problem 3) (10 points) (no justifications are needed)

A function $f(x,y)$ of two variables has level curves as shown in the picture.

Enter A-E	Description
	a critical point of $f(x,y)$.
	a point, where f is extremal under the constraint $x = 2$.
	a point, where f is extremal under the constraint $y = 0$.
	the point among points A-E, where the length of the gradient vector is largest.
	the point among points A-E, where the length of the gradient vector is smallest.
	a point, where $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$ and $D_{\langle 1,-1\rangle\sqrt{2}} f \neq 0$.
	a point, where $D_{\langle 1,-1\rangle/\sqrt{2}} f = 0$ and $D_{\langle 1,1\rangle/\sqrt{2}} f \neq 0$.
	a point, where $D_{\langle 1,0\rangle} f = 0$ and $D_{\langle 0,1\rangle} f \neq 0$.
	a point, where $f_x = 0$ and $f_y \neq 0$.
	a point, where $f_y = 0$ and $f_x \neq 0$.



Solution:

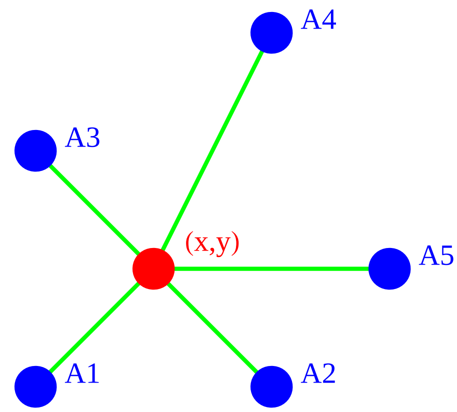
Enter A-E	Description
A	a critical point of $f(x, y)$.
E	a point, where f is extremal under the constraint $x = 2$.
D	a point, where f is extremal under the constraint $y = 0$.
E	the point among points A-E, where the length of the gradient vector is largest.
A	the point among points A-E, where the length of the gradient vector is smallest.
B	a point, where $D_{\langle 1,1\rangle} f = 0$ and $D_{\langle 1,-1\rangle} f \neq 0$.
C	a point, where $D_{\langle 1,-1\rangle} f = 0$ and $D_{\langle 1,1\rangle} f \neq 0$.
D	a point, where $D_{\langle 1,0\rangle} f = 0$ and $D_{\langle 0,1\rangle} f \neq 0$.
D	a point, where $f_x = 0$ and $f_y \neq 0$.
E	a point, where $f_y = 0$ and $f_x \neq 0$.

Problem 4) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.



Solution:
The gradient of f is

$$\nabla f(x, y) = \langle -14 + 10x, -12 + 10y \rangle .$$

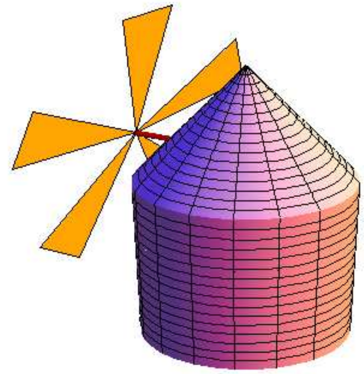
It leads to the solution $(x, y) = (7, 6)/5 = (1.4, 1.2)$.
(Side remark: In general the average $\sum_{i=1}^{n} A_i/n$ is the only critical point because the function $f(X) = \sum_{i=1}^{n} |x - A_i|^2$ has the gradient $\sum_{i=1}^{n} 2(X - A_i) = 0$ showing $nX = \sum_i A_i$. This is true in any dimension and any number of mass points.)
The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} .$$

From this we can read $D = 100$ and $f_{xx} = 10$. The second derivative test shows that the point is a minimum. We have $f(1.4, 1.2) = 14$.

Problem 5) (10 points)

The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius r, the height of the side wall is h and the height of the roof is $\sqrt{3}r$, then the volume is



$$V(h,r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and area of the building is

$$A(h,r) = \pi r^2 + 2\pi rh + \pi 2r^2 = \pi(3r^2 + 2rh) .$$

For fixed volume $V(h,r) = 4\pi/3$, minimize $A(h,r)$ using the Lagrange multiplier method.

Solution:
The Lagrange equations are

$$\begin{aligned} 3*r+h &= \lambda 4hr/3 \\ 2 &= \lambda r/3 \\ r^2h &= 1 \end{aligned}$$

You can plug in λr from the second equation into the first to get

$$\begin{aligned} 3r+h &= 2h \\ r^2h &= 1 \end{aligned}$$

The first equation shows $h = 3r$ and plugging this into the third equation gives $\boxed{r = 1/3^{1/3}}$ and $\boxed{h = 3r = 3^{2/3}}$.

Problem 6) (10 points)

a) (5 points) Find the tangent plane to the surface $x^3y+yx^3+z^2x^2 = 6$ at the point $(1,1,2)$.

b) (5 points) Find the tangent line to the curve $x^4 - y^4 = 15$ at the point $(2,1)$.

Solution:
a) The gradient is

$$\langle 6x^2y + 2xz^2, 2x^3, 2zx^2 \rangle \ .$$

At the point $(1,1,2)$, this is $\langle 14, 2, 4 \rangle$. The equation is $\boxed{14x + 2y + 4z = 24}$.
b) The gradient vector is $\langle 4x^3, -4y63 \rangle$ which is $\langle 32, -4 \rangle$. The equation of the tangent line is $32x - 4y = 60$ or $\boxed{8x - y = 15}$.

Problem 7) (10 points)

a) (4 points) Compute the moment of inertia

$$I = \int \int_G (x^2 + y^2) \; dydx$$

of the half disc $D = \{x^2 + y^2 \leq 1, x \geq 0 \}$.
b) (6 points) Evaluate the following double integral

$$\int_1^e \int_{\log(x)}^1 \frac{y}{e^y - 1} \; dydx \ ,$$

where log is the natural log as usual.

Solution:
a) Use polar coordinates! For right half disc, we have

$$\int_0^1 \int_{-\pi/2}^{\pi/2} r^3 \; d\theta dr = \pi/4 \ .$$

We wanted to see the right integral too.
The result is $\boxed{\pi/4}$.
b) Change the order of integration! It is important here to make a picture! We end up with the integral

$$\int_0^1 \int_1^{e^y} \frac{y}{e^y - 1} \; dxdy \ .$$

Now, the inner integral is no problem and gives y. The result is $\int_0^1 y \; dy = \boxed{1/2}$.

Problem 8) (10 points)

a) (4 points) Find the linearization $L(x, y)$ of the function $f(x, y) = x^5 \cdot y^3$.

b) (6 points) Estimate $10.01^5 \cdot 4.999^3$ using the linear approximation found in part a).

Solution:
We will call this problem the "Who wants to be a millionaire" problem from now on.
a) $\nabla f(x, y) = \langle 5x^4y^3, 3x^5y^2\rangle = \langle 6'250'000, 7'500'000\rangle$. Since $f(10, 5) = 10^5 5^3 = 12'500'000$ so that the linearization is $12'500'000 + 6'2500'000(x - 10) + 7'500'000(y - 5)$.
b) The estimation is $12'500'000 + 62500 - 7; 500 = \boxed{12'555'000}$. This is pretty close to the real value $12'562'600$.

Problem 9) (5 points)

Find the surface area of the surface parametrized by

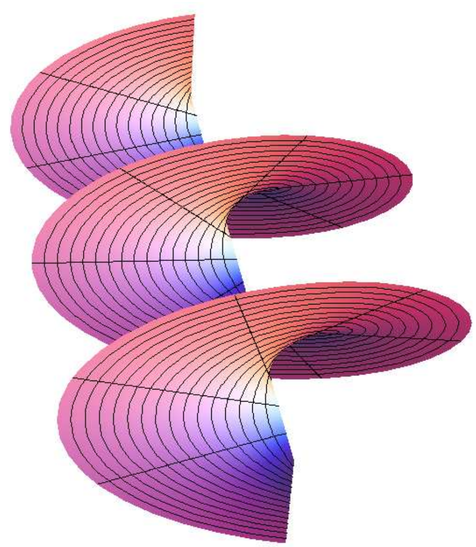
$$\vec{r}(t, s) = \langle s\cos(t), s\sin(t), t\rangle ,$$

where $0 \le t \le 5\pi$ and $0 \le s \le 2$.

Hint. You can use the anti derivative formula

$$\int \sqrt{1 + s^2}\, ds = s\sqrt{1 + s^2}/2 + \operatorname{arcsinh}(s)/2$$

computed in class without having to derive it again.



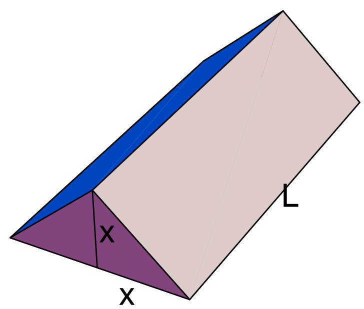
Solution:
The surface area is

$$\int_0^{5\pi} \int_0^2 |\vec{r}_t \times \vec{r}_s| \; dsdt \; .$$

A straightforward computation gives $|r_t \times r_s| = \sqrt{s^2+1}$. We get $\boxed{5\pi(\sqrt{5} + \text{arcsinh}(2)/2)}$.

Problem 10) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x,y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.



Solution:
The Lagrange equations

$$\nabla f = \lambda \nabla g, g = 1$$

are

$$\begin{aligned} 4x + 4y &= \lambda 2xy \\ 4y &= \lambda x^2 \\ xy^2 &= 1 \end{aligned}$$

Eliminating λ gives $(4x+4y)/4x = \lambda 2xy/\lambda x^2$ or $1 + y/x = 2y/x$ so that $1 = y/x$. The only critical point with positive x, y is $(1,1)$. The minimum of f is $f(1,1) = 6$. The minimal surface area is $6\sqrt{2}$.