7/25/2013 SECOND HOURLY PRACTICE VIII Maths 21a, O.Knill, Summer 2013

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

1) T F (F highlighted) It is possible that $(1,1)$ is a local maximum for the function f and $1 = f_{xx} = -f_{yy}$.

Solution:
If $f_{xx} = -f_{yy}$ then

2) T F (T highlighted) $(0,0)$ is a local maximum of the function $f(x,y) = 5 - x^8 - y^8$.

Solution:
$(0,0)$ is a local maximum because the value there is 5 and the function is smaller everywhere else.

3) T F (T highlighted) If the curvature of a curve is zero everywhere, then it is a line.

Solution:
One can see that by the formula but also by physics.

4) T F (F highlighted) If the Lagrange multiplier λ is negative then the critical point under constraint is a saddle point.

Solution:
The sign of λ has nothing to say about the nature of the critical point.

5) T F (F highlighted) The arc length of a curve on $[0,1]$ can be obtained by integrating up the curvature of the curve along the interval $[0,1]$.

Solution:
For a line the curvature is zero but the arc length is not zero.

6) T F (F highlighted) If D is the discriminant at a critical point and $Df_{xx} > 0$ then we either have a saddle point or a local maximum.

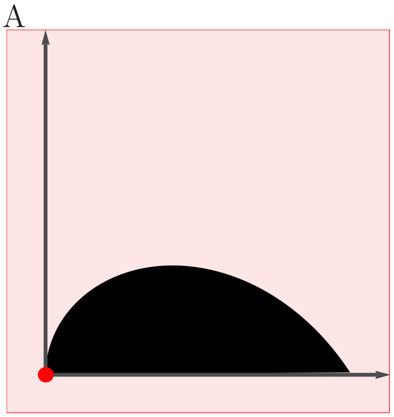
Solution:
It is either a saddle point or a local minimum.

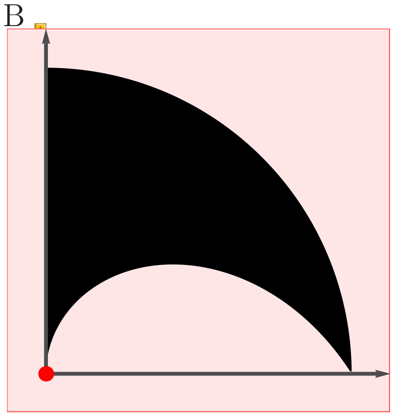
7) T F The function $f(x, y) = \sin(y)x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xyyxyxy} = 0$.

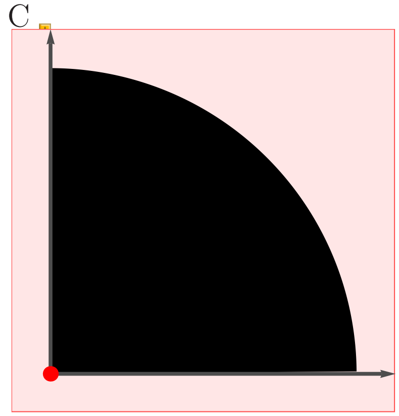
Solution:
By Clairot's theorem, we can have all three x derivatives at the beginning.

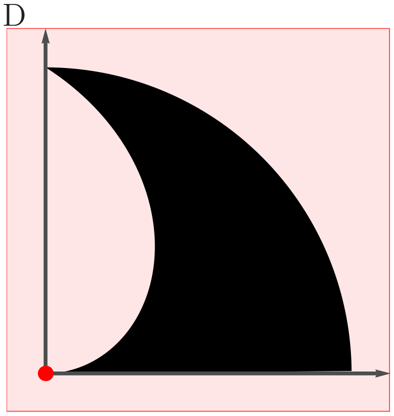
Problem 2) (10 points)

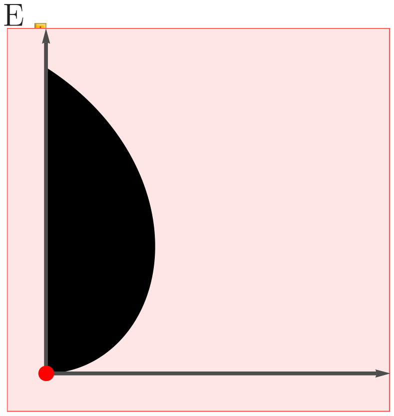
Match the regions with the corresponding polar double integrals

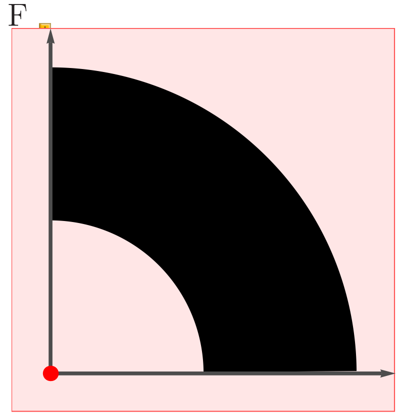
A


B


C


D


E


F


Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta) r \; dr d\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_{\theta}^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta) r \; dr d\theta$

Solution:
C,E,A,D,B,F

Problem 3) (10 points)

The following statements are not complete. Fill in from the pool of words below.

statement	Fill in the letters	statement
The surface area does		on the parametrization.
$\sqrt{48}$ can be estimated by		at $x = 7$. The result is 7-1/14.
The discriminant D is		if the point is a saddle point.
For a Lagrange minimum, ∇g is		to ∇f.
Arc length is approximated by a		sum if the curve is smooth.
The gradient ∇f is		to the surface $f = c$.

O	not depend
L	linear approximation
I	negative
D	not depend
M	tangent
O	parabola
E	perpendicular
L	parallel
E	orthogonal
R	Rieman

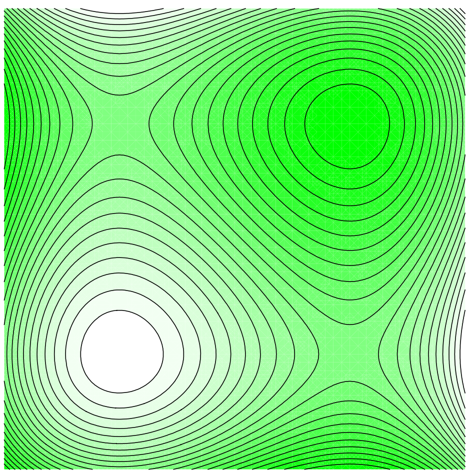
Solution:
OLIVER

Problem 4) (10 points)

The green near one of the holes in the Cambridge Fresh pond golf course has the height

$$f(x, y) = x^3 + y^3 - 3x^2 - 3y^2$$

Find local maxima, local minima or saddle points of this function. Near which point will golf balls most likely end up, if balls like to roll to lower areas.



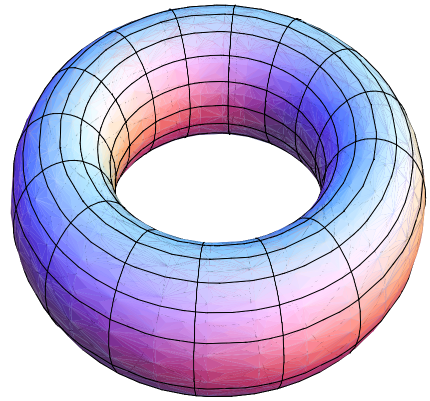
Solution:
$\nabla f(x, y) = -6x + 3x^2, -6y + 3y^2 = (0, 0)$ so that the critical points are $(0, 0), (0, 2), (2, 0), (2, 2)$. We have $D = 36(x - 1)(y - 1)$ and $f_{xx} = 6x - 6$.

Point	D	f_{xx}	type
$(0, 0)$	$D = 36$	-6	max
$(0, 2)$	$D = -36$	-6	saddle
$(2, 0)$	$D = -36$	6	saddle
$(2, 2)$	$D = 36$	6	min

The ball more likely ends up near the minimum $(2, 2)$.

Problem 5) (10 points)

A torus can be obtained by rotating a circle of radius b around a circle of radius a. The volume of such a torus is $2\pi^2 ab^2$ and the surface area is $4\pi^2 ab$. If we want to find the torus which has minimal surface area while the volume with fixed packing $2\pi^2 a(b^2+1)$ is fixed $2\pi^2$, we need to extremize the function $f(a,b) = 4\pi^2 ab$ under the constraint $a + ab^2 = 1$. Find the optimal a, b.



Solution:
The Lagrange equations $\nabla f(a,b) = \lambda \nabla g(a,b), g(a,b) = 1$ are

$$\begin{aligned} 4\pi^2 b &= \lambda(b^2+1) \\ 4\pi^2 a &= 2\lambda ab \\ a + ab^2 &= 1 \end{aligned}$$

Dividing the first equation by the second gets rid of λ and leavs the two equations for the unknown a, b:

$$\begin{aligned} b/a &= (b^2+1)/(2ab) \\ a + ab^2 &= 1 \end{aligned}$$

The first equation is $ab^2 = a$ and since the second equation excludes $a = 0$ we have $b^2 = 1$ or $b = 1$ ($b \geq 0$ because it is a radius). The last equation gives $a = 1/2$.

Problem 6) (10 points)

a) Find the arc length of the curve $\vec{r}(t) = \langle t^2, 2t^3/3, 1\rangle$ from $t = -1$ to $t = 1$.
b) What is the curvature of the curve at time $t = 1$? The formula for the curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r\,'}(t)|^3}$$

Solution:
a) $r\,'(t) = \langle 2t, 2t^2, 0\rangle$ and $|r\,'(t)| = \sqrt{4t^2 + 4t^4} = 2|t|\sqrt{1+t^2}$. Note the absolute value because the velocity is always nonnegative.

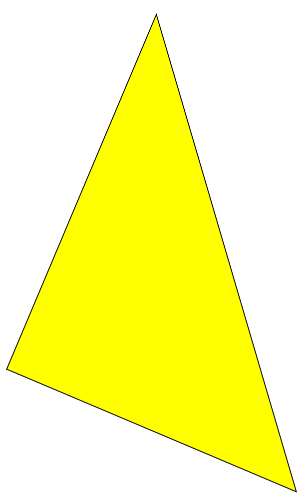
$$\int_{-1}^{1} 2|t|\sqrt{1+t^2}\,dt = 2\int_0^1 2t\sqrt{1+t^2}\,dt = \frac{4}{3}(1+t^2)^{3/2}|_0^1 = 4(\sqrt{2}^3 - 1)/3\,.$$

b) $r\,'(1) - \langle 2, 2, 0\rangle$ and $r\,'(1) = \langle 2, 4, 0\rangle$ so that

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r\,'}(t)|^3} = \frac{|\langle 0, 0, 4\rangle|}{\sqrt{8}^3} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}\,.$$

Problem 7) (10 points)

A right angle triangle has the side lengths $x = 0.999$ and $y = 1.00001$. Estimate the value of the hypotenuse $f(x, y) = \sqrt{x^2 + y^2}$ using linear approximation.



Solution:
We estimate near $(x_0, y_0) = (1, 1)$. The linearization function $L(x, y)$ is defined as

$$L(x, y) = f(1, 1) + \nabla f(1, 1) \cdot \langle x - 1, y - 1 \rangle$$

Now $\nabla f(x, y) = \langle x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2} \rangle$ and $\nabla f(1, 1) = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ we have

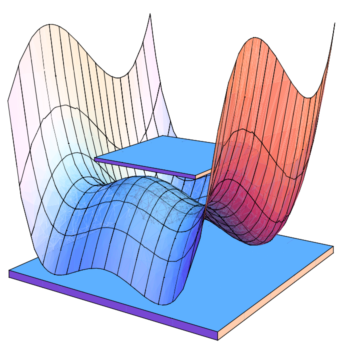
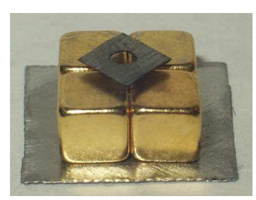
$$L(x, y)\sqrt{2} + (x - 1)/\sqrt{2} + (y - 1)/\sqrt{2} \ .$$

For the values given, we have $x - 1 = -0.001, y - 1 = 0.00001$ and so the estimate

$$\sqrt{2} - 0.001/\sqrt{2} + 0.00001/\sqrt{2} \ .$$

Problem 8) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form $f(x, y) = x^4 + y^3 - 2x^2 - 3y$. Find all critical points of this function and classify them. Is there a global minimum?

Right picture credit: Wikipedia.

Solution:

To find the critical points, we have to solve the system of equations $f_x = 4x^3 - 4x = 0, f_y = 3y^2 - 3 = 0$. The first equation gives $x = 0$ or $x = \pm 1$. The second equation $f_y = 3y^2 - 3 = 0$ gives $y = \pm 1$. There are $3 \cdot 2 = 6$ critical points. We compute the discriminant $D = 6y(12x^2 - 4)$ and $f_{xx} = 12x^2 - 4$ at each of the 6 points and use the second derivative test to determine the nature of the critical point.

point	D	$f_x x$	nature	value
(-1, -1)	-48	8	saddle	1
(-1, 1)	48	8	min	-3
(0, -1)	24	-4	max	2
(0, 1)	-24	-4	saddle	-2
(1, -1)	-48	8	saddle	1
(1, 1)	48	8	min	-3

There is no global minimum, nor any global maximum since for $x = 0$, the function is $f(0, y) = y^3 - 3y$ which is unbounded from above and from below (it goes to $\pm\infty$ for $y \to \pm\infty$).