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LETTER TO THE EDITOR

On an Episode in the History of the Integral Calculus

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A review of Professor Judith Grabiner's *The Origins of Cauchy's Rigorous Calculus* in *Historia Mathematica* [Dhombres 1985] reflected the current state of knowledge of the first great revolution in mathematical rigor since the time of Euclid. Inasmuch as there is a received view and a nonstandard historiography [Robinson 1966] in the abundant secondary literature, but not, in my opinion, a consistent explanation of this important episode in modern foundations, mathematical knowledge is still to be studied in its making. I should like in this short note to make a few comments on the origins of Cauchy's program [Grabiner 1981, 77–113].

The Cauchy Enigma

Following Grabiner's analysis [Grabiner 1978; Flett 1974] of the Amperian definition of the derivative by "exhaustion," I undertook a reading program in the archives of the Ecole Polytechnique and the Académie des Sciences, hoping to find evidence related to our subject in several places. Cauchy's activity was a very short intellectual adventure from 1815 to 1821 which can be traced with minute fine-grained chronology in the logbooks entitled "registres de l'instruction," preserved in the archives of the Ecole Polytechnique [Grattan-Guinness 1986], and in several direct sources. For example, I found a manuscript in Ampère's *Nachlass* (Institut de France) and a notebook in the house of the French inventor of sociology, Auguste Comte [Guitard 1986], and also lesser-known printed matter such as Ampère's lectures in 1822 (one example in the Sorbonne library of the University of France, Paris).

Ampère's Ontological Proof of 1806

In the second chapter of her book Grabiner [1981, 28] emphasized the persistence of the Greek tradition and in her conclusion showed how Cauchy's pattern of proof contrasted with the Baconian style of Lacroix' pedagogy, but contained no thorough study of the calculus of limits such as Ampère, who used a concept of "proximité," could read in the posthumous works of Simson [1776]. The Amperian scheme, contemporary with the Peyrard translations, is not the same pattern of mathematical thought as Lagrange's demonstration [Ovaert 1976]; it is a

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dichotomy procedure and not a division process. Ampère used the universal chord theorem (a Cauchian version is in *Comptes Rendus* of 1840), and the so-called Amperian lemma of the "principe fondamental du calcul différentiel et intégral" was, apart from the use of distributive quantities, cast in an implicit Archimedean frame: the thought experiment of *rolling without slipping* underlay the conceptual apparatus at work on the problem of arc length and on the axiom of the ratio of arc to chord [Grabiner 1981, 138–139; Seidenberg 1982; Stolz 1881]. We have here several examples of infinite processes typical of the prehistory of the Borel lemma. The *ut unum* principle of Cavalieri, used by L'Huillier in the calculus of "Lim," was emphasized by Moigno as being "un principe qu'on doit appeler fondamental," and a good construction in Cauchy's language of means is a text of 1841: "mémoire sur le rapport différential de deux grandeurs qui varient simultanément" [Medvedev 1983].

It would be appropriate to discuss the role of the "accroissements finis" theorem in existence and uniqueness proofs [Aziz & Diaz 1963], as it comes after the changes of the 1820s and Fourier's publication on Newton's method. In 1817, Cauchy began his series of lectures on *algebraic analysis* by the "general lemma" [Grabiner 1981, 169n.*]; he used the theory of means (what I call the M-language) at the end of the year in solving differential equations of the form $dy = f(x)dx$ and $dy = yf(x)dx$ [Guitard 1986]. The most surprising fact is that I have never seen a reference to page 444 of Cauchy's *Analyse algébrique* (inegalité fondamentale du théorème d'existence).

Cauchy's Paradigm; Existence Theorems

Actually, the origin of Cauchy's theory of the integral lies in his researches in the theory of ordinary differential equations from 1814; it is historically dangerous not to deal with the development of a subject on which Cauchy worked in the first decade of his mathematical career [Grabiner 1981, Index]. The turning point of Cauchy's analysis was his lost memoir of May 1816 on "solutions particulières" (Laplace), described after 1817 as singular integrals, a very Lagrangian subject, as it was the purpose of the "calcul des fonctions" [Lagrange 1806] to give a solution to the problem. So my main interest in finding Comte's notebook and Ampère's lectures was in trying to reconstruct this epoch-making memoir, which was the origin of a subsequent polemic with Poisson in 1821 (on the singular solution of $y' = y(\ln(y))^a$).

Objectively speaking, Cauchy used the "modulus of continuity" as a condition for uniqueness of Osgood type for $dy = f(x, y)dx$. This criterion initiated the reconstruction of the integral calculus, of which the second step was the lectures at the Collège de France of 1817. In the 1817 lectures on equations, continuity was used as a condition for the convergence of certain approximation procedures, because Cauchy considered that mathematicians had rightly regarded the question of whether every equation has a root as fundamental [Grabiner 1981, 155]. Cauchy's first published proof in analysis was "his" proof of the *fundamental* *theorem of algebra* of October 1817, but similarly there was another fundamental question: the existence of solutions of differential equations.

The Locus Classicus of the New Analysis: Lesson VI of 1824 [*Gilain 1981*]

In 1817, at the same time as Ampère introduced the "factorisation des fonctions entières de degré infini" in the Polytechnique, Cauchy, in December, studied the equation $dy = yf(x)dx$, which he solved by an *integral product*, using a limiting process and continuity to pass from the finite to the infinite. In the Eulerian product, and the converse Fermat integral sum, Cauchy found the same ballistic method which presumed a discretization of time in the concept of the differential ratio (*hapax* of 1823). The question is to understand the lessons in the logbook and their unique published occurrence. There was a change which was linked to the development of new techniques: he checked a criterion for singularities after creating a new theory of tangency (see below) and put his constructive proof of the fundamental theorem of algebra onto a sound basis, which then became paradigmatic for all his lectures.

One must insist on his use of *Newton's algorithm* as the origin of the second proof of the existence of solutions to $dy = f(x, y)dx$ [Gilain 1981] after the new course in 1820 ("détermination des erreurs que l'on peut commettre dans l'intégration par approximation," December 1821) because the books on Cauchy fall too short in their treatment of gradient methods and there are a number of papers by Cauchy on numerical methods [Goldstine 1977; Grabiner 1981, 57]. Grabiner stressed the Lagrangian legacy, but her guideline is the story of a disorderly retreat. Cauchy initiated a new strategy after he had learned tactics in the works of a good soldier, Legendre, borrowing from him the Argand proof (1808), table making and means (1815), *criticism of Lagrange's expansions in the reduction of the ecliptic* as approximation procedures if not *a priori* proofs [Grabiner 1981, 149]. Legendre was addicted to Newton's method (see his 1808 proof and his supplement of February 1816).

Taylor's Formula and Lagrange's Series

In the "Résumé" Taylor's formula was used to solve differential equations, with special attention paid to initial conditions. We can see here a protohistory of the Peano series (see the integral product), not a technique of Picard "itérants" (see the integration of $d^ny/dx^n = f(x)$ and $d^ny/dx^n = yf(x)$ in the "table des matières" for the years 1827–1828). One more comment on these series is in order: the theory of extrema was deduced from a preparatory theorem concerning the division of functions, a problem in the theory of tangencies, factorizations, and a second criterion, concerned with a search for exceptions and monsters (contacts of infinite order and oscillating factors) [Grabiner 1981, 137].

Algebraic Analysis from Vieta to Cauchy, 1817 [*Dhombres* 1985, 88]

Apparently it is a common view today that in 1820 Cauchy wrote a dissertation on the binomial theorem, but, as was clearly stated in the "preface," the "cours d'analyse" was organized to give proofs of factorization theorems for entire functions and to refute the Eulerian "généralité de l'algèbre" as it had been applied to trigonometrical expansions and transcendental equations. In his first lectures of 1817, Cauchy studied the Eulerian problem of the discontinuities of x^y (this became the theory of [1821, Chap. VII, 227]—the study of 0° dates from 1820 but the use of a functional equation as a tool in interpolation dates from 1818); and he also studied D'Alembert's "principle of permanence of form," submitting each step of the Argand–Legendre proof to criticism and counterexamples. A central topic was that of "sections angulaires" [Lagrange 1806, Chap. 11; Legendre 1811], for there was a problem of the infinite here and the equality of Taylor's series, but there was also the problem of the use of singularities of entire functions in recurrent series. So the centerpiece of the new analysis was to be a theory of complex logarithms derived without the aid of the integral calculus (the arithmetization of 1817), in which the binomial theorem furnished a tool for the reversion of series (the problem of Lagrange and Burmann). In a manuscript I found in Cauchy's *Nachlass*, Cauchy, writing for the instruction of his friend Ampère, sketched his main results:

(1) the existence proof for $dy = f(x)dx$;

(2) the fundamental theorem of analysis with a so-called Lipschitz condition, and the discussion of the variational problem (dependence on initial conditions) using a discrete propagator;

(3) the elements of complex logarithms used to establish one of the series of considerable interest in astronomy [1821, Chap. XIX, formula 38], a difficulty mastered in 1817;

(4) lectures on rectilinear triangles in February and a proof in March following the fundamental theorem of algebra, with the same technical problem as "the argument principle."

This new approach is, in my opinion, a major improvement in our understanding of Cauchy, and it would be fair to add that for Cauchy scholars, the game is afoot!

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