8/7/2014 FINAL EXAM **Maths 21a, O. Knill, Summer 2014**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-2 do not require any justifications. Problem 3 only 1-2 words. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The equation $G_{xx}G_{yy} - G_{xy}^2 = 1$ is an example of a partial differential equation

Solution:
Yes

2) T F The curvature of a curve is $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|$.

Solution:
There is no power 3 in the denominator

3) T F If S is the boundary of a solid E and $\int\int\int_E 1 \; dV = 1$, then $\int\int_S |r_u \times r_v| \; dudv = 1$.

Solution:
This is already wrong for the sphere normalized so that it has unit volume. Its also false for the unit cube, where the volume is 1 but the surface area is 6.

4) T F The cross product of two nonzero perpendicular vectors has length 0.

Solution:
It is the product of the lengths.

5) T F The function $f(x, y, z) = |\text{curl}(\vec{F}(x, y, z))|$ is is maximal, where the divergence of $\vec{F}$ is zero.

Solution:
This is random nonsense.

6) T F The distance between two circles of radius 1 in space is equal to the distance between the centers plus 2.

Solution:
If the two circles are in parallel planes, then the distance is equal to the distance of the centers.

7) | T | F | For an ellipse contained in the xz-plane, the binormal vector $\vec{B}$ is always perpendicular to that plane.

Solution:
Yes, the vectors T, N are both parallel to the plane.

8) | T | F | The surface $x^2 - y^2 - z^2 = 1$ is a one sheeted hyperboloid.

Solution:
It is a two sheeted hyperboloid.

9) | T | F | Let $G = \text{curl}(\vec{F})$. A flow line of the vector field $\vec{G}$ is always a circle.

Solution:
We have seen an example where $\text{curl}(\vec{F} = \langle 0, 0, 1\rangle$, for which the flow line is a line.

10) | T | F | The cross product of two unit vectors $\vec{v}$ and $\vec{w}$ has length ≤ 1.

Solution:
This follows from the formula.

11) | T | F | The unit tangent vector $\vec{T}'(t)$ always has length 1.

Solution:
We normalize it to have length 1.

12) | T | F | The discriminant $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ is always smaller than the length of $\nabla f(x, y)$ at the point.

Solution:
Nonsense

13) | T | **F** | The curl of an incompressible vector field is zero.

Solution:
No, it is the divergence which is zero.

14) | **T** | F | $\vec{r}(u,v) = \langle u^3, 2, u^3 + v \rangle$ parametrizes a plane.

Solution:
Yes

15) | T | **F** | The integral $\int\int\int_E |\text{curl}(\vec{F}(x,y,z))| \, dxdydz$ is zero for all solids E.

Solution:
No

16) | **T** | F | If two flow lines of a vector field $\vec{F}$ intersect at a nonzero angle, then the vector field $\vec{F}$ is zero at that point.

Solution:
We have $\vec{r}'(t) = \vec{F}(\vec{r}(t))$. If the curve intersect, then we can not have two different vector fields.

17) | T | **F** | For any vector field $\vec{F}$ we have $\text{div}(\text{grad}(\text{div}(\vec{F}))) = 0$ at every point.

Solution:
Why should it be.

18) | T | **F** | The triple scalar product is always positive.

Solution:
Take $\vec{v} = \vec{w}$.

19) T F If $\vec{v} = \nabla f/|\nabla f|$ at a non-critical point, then the directional derivative of f in the direction of $\vec{v}$ is positive.

Solution:
Yes, this is an important formula

20) T F The vector projection $\vec{P}_{\vec{v}}(\vec{w})$ is a vector which has length smaller or equal than the length of the vector $\vec{w}$.

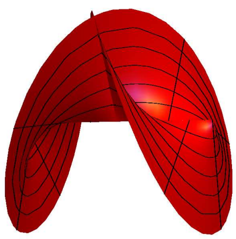
Solution:
we see it from the explicit formula $|\vec{v} \cdot w|/|\vec{v}| \leq |w|$.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the following surfaces. There is an exact match.

Surface	Enter 1-4
$\vec{r}(u,v) = \langle \cos(u), \cos(v), \sin(u) + v \rangle$	
$\vec{r}(t,s) = \langle 4\sin(t), s\sin(2t), s\cos(2t) \rangle$	
$\vec{r}(t,s) = \langle s\cos(t), s\sin(t+s), s \rangle$	
$x^4 + y^4 + z^4 - 6x^2 = 0.3$	

1



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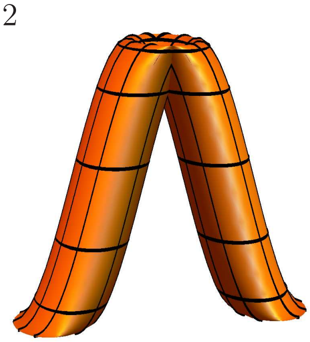

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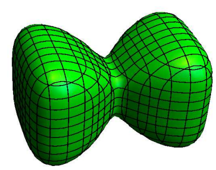

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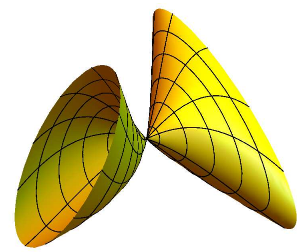


b) (3 points) Match the integral types with the names. There is an exact match.

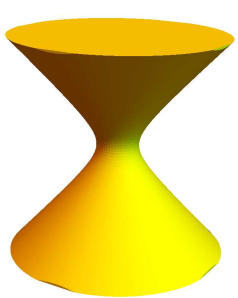
Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert\ dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$	
$\int_a^b \int_c^d \lvert\vec{r}_u \times \vec{r}_v\rvert\ dudv$	

A	line integral
B	flux integral
C	arc length
D	surface area

c) (3 points) Match the solids. There is an exact match.

Solid	Enter a-d
$\lvert xyz\rvert \leq 1$	
$x - y^2 + z^2 \leq 1$	
$x^2 + y^2 - z^2 \leq 1$	
$1 \leq x^2 + y^2 \leq 2$	

a



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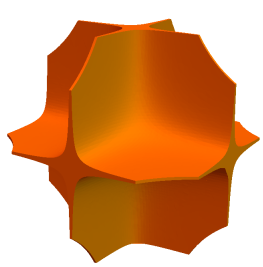
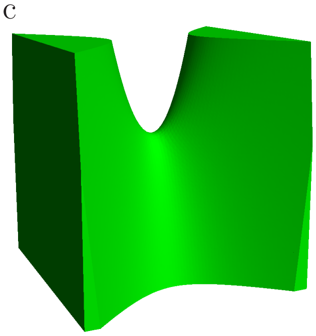
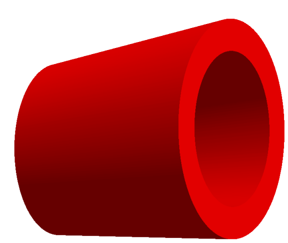


c



d



Solution:
The first problem a) turned out to be a bit tricky. Matching 3) was no problem but for the others one has to look at the structure of grid curves. Problems b) and c) were very well solved.
a) 2,1,4,3
b) B,C,A,D
c) b,c,a,d

Problem 3) (10 points) Only one word justifications necessary

Physics Nobel prize winner **Richard Feynman** (we have seen him in a scene of the movie "Infinity" in class), has written a book called "6 easy pieces". Inspired by the title, we write 6 problems:

a) (1.666 points) What is the flux of $\vec{F} = \langle x, 2y, 3z \rangle$ through the unit sphere $x^2 + y^2 + z^2 = 1$, oriented outwards?

Answer:	because of:

b) (1.666 points) What is double integral $\int \int_G \text{curl}(\vec{F})\ dxdy$, where G is the unit disk $\{x^2+y^2 \leq 1\}$ and $\vec{F} = \langle -3y, 2x \rangle$?

Answer:	because of:

c) (1.666 points) What is the flux of the curl of the vector field $\vec{F} = \langle x^2, y^3, z \rangle$ through the surface $x^2 + y^4 + z^6 = 1$ oriented outwards?

Answer:	because of:

d) (1.666 points) What is the line integral of $\vec{F} = \langle x^3, y^5, z^6 \rangle$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(3t) \rangle$ parametrized from $t = 0$ to $t = 2\pi$?

Answer:	because of:

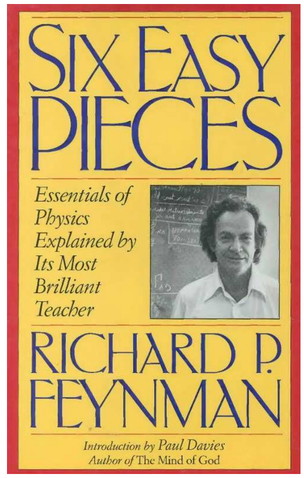
e) (1.666 points) What is the flux of the curl of the vector field $\vec{F} = \nabla f$ through the upper ellipsoid $x^2/4 + y^2/9 + z^2/25 = 1, z \geq 0$?

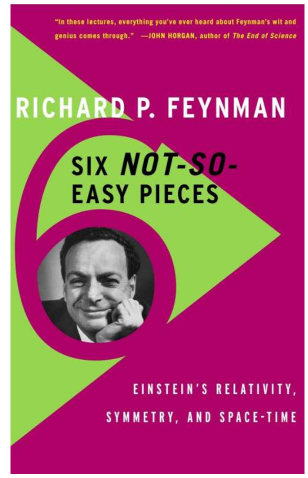
Answer:	because of:

f) (1.666 points) What is the line integral of the vector field $\vec{F}(x, y) = \langle 1, 0 \rangle$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ from $t = 0$ to $t = \pi$?

Answer:	because of:

Why use fractional points? First of all, because it is more fun to grade, second because fundamental particles like quarks also can have fractional charge. We will use the formula $P = \min(10, 2C)$ for the points P earned in this problem, where C is the number of correct answers.
By the way, there is an other book coauthored by Feynman called "Six not so easy pieces". That might also apply to the problems above.

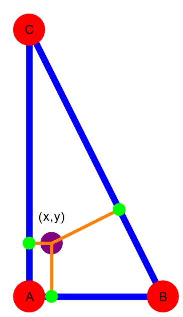




Solution:
a) $6 \times 4\pi/3 = 8\pi$ by the divergence theorem.
b) 5π by Green's theorem
c) 0 by the divergence theorem. It also follows from Stokes theorem
d) 0 by the fundamental theorem of line integrals as we have a closed loop. e) 0 because $\text{curl}\nabla f = 0$ everywhere.
f) -2 by the fundamental theorem of line integrals and using the potential $f(x, y) = x$.

Problem 4) (10 points)

Given the triangle $A = (0, 0, 0), B = (1, 0, 0), C = (0, 2, 0)$, we aim to find the point $P = (x, y, 0)$ for which the sum of the squares of the distances to the triangle edges is minimal.



a) (3 points) Give a formula for the distance of $P = (x, y, 0)$ to the line through the points B and C.

b) (7 points) The sum of the distance squares multiplied by 5 turns out to be

$$f(x, y) = 5x^2 + 5y^2 + (2x + y - 2)^2 .$$

Find the critical points of $f(x, y)$ and classify them.

Solution:
a) The distance from a point $P = (x, y, 0)$ to line B, C is $PB \times BC/|\vec{BC}| = |\langle x-1, y, 0\rangle \times \langle -1, 2, 0\rangle|/\sqrt{5} = |-2 + 2x + y|/\sqrt{5}$.
b) Solving $\nabla f(x, y) = \langle 0, 0\rangle$ gives $(2/5, 1/5)$. There is only one critical point $(2/5, 1/5)$. We see $D = 500$ and $f_{xx} = 18$ so that this is a minimum.

Problem 5) (10 points)

We all adored “monkey” in the movie “Hangover II”. On the picture you see him sitting on his **Monkey saddle** $z = f(x, y) = x^3 - 3xy^2$ which is called ”Phil”.

a) (4 points) Find the tangent plane to $g(x, y, z) = z - f(x, y) = 0$ at $(2, 1, 2)$.

b) (4 points) Estimate $2.001^3 - 3 \cdot 2.001 \cdot 0.99^2$.

c) (2 point) Find the directional derivative $D_{\vec{v}} f$ at $(2, 1)$ if $\vec{v} = \langle 0, 1 \rangle$.

Solution:
a) We have the surface $z - x^3 + 3xy^2 = 0$. The gradient is $\langle -3x^2 + 3y^2, 6xy, 1 \rangle$ which is $\langle -9, 12, 1 \rangle$ at $(2, 1, 2)$. The equation of the plane is $-9x + 12y + z = d$ where $d = -4$ can be obtained from plugging in the point $(2, 1, 2)$. The equation is $-9x + 12y = z = -4$.
b) We have $\nabla f(x, y) =)3x^2, -6xy)$ and $\nabla f(2, 1) = \langle 9, -12 \rangle$. The linearization is $L(x, y) = 2 + 9(x - 2) - 12(y - 1)$ and estimate $L(2.001, 0.99) = 2 + 9 \cdot 0.001 - 12(-0.01) = 2.129$.
c) As $\nabla f(2, 1) = \langle 9, -12 \rangle$ we have $D_v f = \langle 9, -12 \rangle \langle 0, 1 \rangle = -12$.

Problem 6) (10 points)

This is the last chance to show that you are not a zombie!

a) (5 points) Evaluate the double integral

$$\int_0^3 \int_{x^2}^9 \frac{x}{e^{y^2}} \, dy dx \, .$$

b) (5 points) Integrate the curl of $\vec{F}(x, y) = \langle -yx^2, xy^2 \rangle$ over the region $4 \leq x^2 + y^2 \leq 9$.

Solution:
a) Change the order of integration:

$$\int_0^9 \int_0^{\sqrt{y}} \frac{x}{e^{y^2}} \, dxdy = -\frac{1}{4} e^{-y^2} |_0^9 = \frac{1 - e^{-81}}{4} \; .$$

It can also be written as $\boxed{(e^{81} - 1)/4}$.

b) The curl is $x^2 + y^2$. Integrating gives $\int_0^{2\pi} \int_2^3 r^3 \, drd\theta = 2\pi(81 - 16)/4 = \boxed{65\pi/2}$. It was also possible to use Green and compute the line integral instead, but that was harder. As expected, the integration factor r was often forgotten.

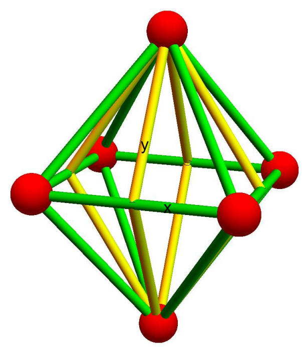
Problem 7) (10 points)

A **diamond** has the shape of an octahedron. Assume the side length of the equator is x and the side height y. Extremize the surface area

$$f(x, y) = 4xy$$

of the diamond under the constraint that the total length of the supporting height pieces is constant

$$g(x, y) = 4x + 8y = 16 \; .$$



Solution:
This is a typical Lagrange problem. The Lagrange equations $4y = \lambda 4, 4x = \lambda 8$ give after elimination $x = 2y$ and plugging this in gives $x = 2, y = 1$. While grading, we did not insist the other critical $(0, 2), (4, 0)$ to be seen. If you analyze the solution, you see that the extremal cases are all geometrically special leading to octahedra of zero volume. Geometrically more interesting related problems use the volume of V, but those Lagrange problems are harder as they involve square root expressions needing some creative manipulations to be solved without computer.

Problem 8) (10 points)

Yesterday on August 6th, 2014, the **Rosetta space craft** arrived at the 2.5 miles wide Churyumov-Garasimenko comet. This coming November, a lander **Philae** will set down on the comet by harpooning itself to the surface and becoming the first space craft landing on a comet. Assume the deceleration is $\vec{r}\,''(t) = \langle -t, -t^2, 0.1 \rangle$ which is a combination of thruster force and microgravity and that $\vec{r}\,(0) = \langle 3, 4, 3 \rangle$ and $\vec{r}\,'(0) = \langle 0, 0, 1 \rangle$. Find the path $\vec{r}(t)$ of the space craft.

Image Credit: European Space Agency: http://www.esa.int

Solution:
Just integrate twice and make sure the constants are right. $\vec{r}(t) = 3 - t^3/3, 4 - t^4/12, 3 - t^2/2 + t$.

Problem 9) (10 points)

Warka water towers are a recent invention. They can pull gallons of fresh water out of thin air. Assume the vector field of the surrounding air is $\vec{G} = \text{curl}(\vec{F})$, where

$$\vec{F}(x, y, z) = \langle x + \sin(z), -2y + z^5 \sin(5z), z \rangle \ .$$

Find the flux of $\vec{G}$ through the surface of revolution given as $S : r(z) = 2 + \sin(z)$ and $0 \leq z \leq \pi$. Note that the surface does not include the bottom. You compute the amount of fluid gathered by the tower in a day.

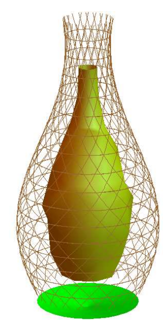


Image credit for first picture: http://www.smithsonianmag.com.

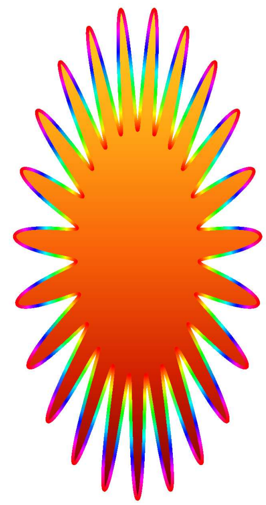
Solution:
This is a Stokes problem. The flux of the vector field through the surface is the sum of the line integrals of $\vec{F}$ along the boundaries: $\vec{r}_1(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$ and $\vec{r}_2(t) = \langle 2\cos(t), -2\sin(t), \pi \rangle$. We have $\vec{F}(\vec{r}_1(t)) = \langle 2\cos(t), -2\sin(t), 0 \rangle$ and $\vec{F}(\vec{r}_2(t)) = \langle 2\cos(t), 2\sin(t), \pi \rangle$. The line integrals are both zero.

Problem 10) (10 points)

Use an integral theorem to find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle (3 + \sin(23t)) \cos(t), 2(3 + \sin(23t)) \sin(t) \rangle$$

from $t = 0$ to $t = 2\pi$. As seen several times in class, you can use that integrating an odd 2π periodic function from 0 to 2π is zero.



By the way, if you stare for an hour at the picture with your nose 3 inches from the paper, you get hallucinations as the curve is an example of a **hypnagogic stimulus curve***. It only works with "23". It can also be used as a* **Rorschach test***: what do you see when you look at the inkblot?*

Solution:
This is a typical Green problem. Use $\vec{F} = \langle 0, x \rangle$ and compute the line integral along the boundary. Use $\vec{r}'(t) = \langle ..., 2(3+\sin(23t))\cos(t) + (3+23\cos(23t))\sin(t) \rangle$. We get the integral

$$\int_0^{2\pi} (3+\sin(23t))\cos(t)[2(3+\sin(23t))\cos(t) + (3+23\cos(23t))\sin(t)] \; dt \; .$$

Using the suggestion in the problem, most terms go away. What remains is

$$\int_0^{2\pi} 18\cos^2(t) + 2\cos^2(t)\sin^2(23t) \; dt \; .$$

Now we use our beloved double angle formulas $\cos^2(t) = (1+\cos(2t))/2$ and $\sin^2(t) = (1-\cos(2t))/2$ for the squares of trig functions and use that integrating $\cos(2t)$ or $\cos(24t)$ from 0 to 2π is zero

$$\int_0^{2\pi} 18/2 + 1/2 - \cos(2t)\cos(46t)/2 \; dt = 19\pi - \int \cos(2t)\cos(46t)/2 \; dt$$

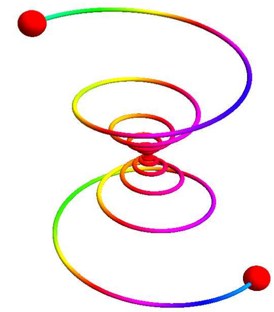
The last integral is zero because $g(t) = \cos(2t)\cos(46t)/2$ has the property that $g(\pi/2+x)$ is an odd function, so that it disappears. The answer is $\boxed{19\pi}$.
Remark The last step was not easy and only about 10 percent of the students could take the last hurdle during the exam without access to material or computers.
P.S: there is a secret weapon here which we do not use in these courses but which allows to solve virtually any trig difficulty without access to computers or formulas: it is by writing $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/(2i)$. Then one has only to know that integrating e^{inx} from 0 to 2π is zero for every n different from zero. But Pssst! Keep it to yourself and don't tell anybody!
P.P.S. When writing the problem originally, the curve really had been hynagogic and crazy. But the complexity of the computations had been 23 times bigger too and after Oliver got sober again from staring at graphs, he simplified the problem. It turned out still to be the computatiolly hardest in this test.
P.P.P.S. The number 23 is everywhere (see the movie!) Look at the date of this exam for example and multiply $8 * 7 * 2014 = 112784$. Guess what the sum of the digits of the result is? 23. What else?

Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x,y,z) = \langle 2x, 3y^2, 3z^3 \rangle$ along the **coil** curve

$$\vec{r}(t) = \langle t\cos(3\pi\log(|te|)), t\sin(3\pi\log(|te|)), t \rangle$$

from $t = -1$ to $t = 1$, where as usual, log is the natural logarithm and $e = \exp(1)$ is the Euler's mathematical constant.

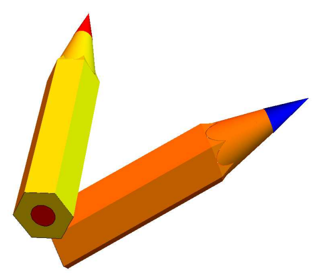


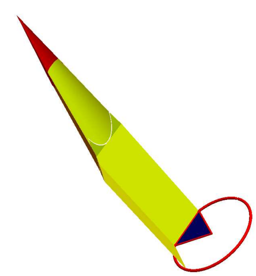
By the way, this curve has finite length but dances around the z-axes infinitely often. Great idea for a roller coaster!

Solution:
We use that the vector field is a gradient field with potential $f(x,y,z) = x^2 + y^3 + 3z^4/4$. We only have to find the beginning and end points $B = \vec{r}(1) = (-1,0,1)$ and $A = \vec{r}(-1) = (1,0,-1)$ and get the result $f(B) - f(A) = 7/4 - 7/4 = 0$.

Problem 12) (10 points)

When Oliver learned calculus he was given the problem to find the volume of the **pencil** E, a hexagonal cylinder of radius 1 above the xy-plane cut by a sharpener below the cone $z = 10 - x^2 - y^2$. Still having nightmares about this, he needs therapy to get rid of this "pencil sharpener phobia": we consider one sixth of the pen where the base is the polar region $0 \leq \theta \leq 2\pi/6$ and $r(\theta) \leq \sqrt{3}/(\sqrt{3}\cos(\theta) + \sin(\theta))$. The pen's back is $z = 0$ and the sharpened part is $z = 10 - r^2$. Write down the triple integral for the volume of E. You get full credit for writing down the correct integral.





P.S. If you have spare time, try to solve the integral but it is probably the "hardest math problem in the world. If somebody of you can solve it, I will make sure that 'none of you will ever have to open a math book again'." (As you know, this line is stolen from the movie "Rushmore".)

Solution:
This is straghtforward as we have a triangular base already given in polar coordinates and the ground $z = 0$ to the roof $z = 10 - r^2$. The integral is

$$\int_0^{\pi/3} \int_0^{\sqrt{3}/(\sqrt{3}\cos(t)+\sin(t))} \int_0^{10-r^2} 1 \; r \; dzdrd\theta \; .$$

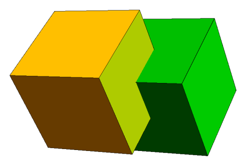
Problem 13) (10 points)

A computer can determine the volume of a solid enclosed by a triangulated surface by computing the fluxe of the vector field $\vec{F} = \langle 0, 0, z \rangle$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = \langle 0, 0, z \rangle$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2\} \;\cup$$

$$\{1 \leq x \leq 3, 1 \leq y \leq 3, 1 \leq z \leq 3\} \; ,$$

where $\cup$ is the union and the surface is oriented outwards.



Ceci n'est pas une pipe
Ceci c'est une vache

Solution:
The divergence is 1. The volume of the cubistic cow is $2^3 + 2^3 - 1 = 15$. The divergence theorem gives the answer 15.