Name:

- Start by writing your name in the above box.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) **T** F   The vector projection of $\langle 2, 3, 4 \rangle$ onto $\langle 1, 0, 0 \rangle$ is $\langle 2, 0, 0 \rangle$.

Solution:
Apply the formula. Because the vector on which we project has length 1, the result is the dot product times this vector.

2) T **F**   The triple scalar product between three vectors is zero if and only if two of the vectors are parallel.

Solution:
They can be nonparallel but in the same plane.

3) **T** F   There are two vectors $\vec{v}$ and $\vec{w}$ so that the dot product $\vec{v} \cdot \vec{w}$ is equal to the length of the cross product $|\vec{v} \times \vec{w}|$.

Solution:
Take two vectors which make an angle of 45 degrees. Then $\sin(\theta) = \cos(\theta)$.

4) **T** F   The distance between two spheres of radius 1 whose centers have distance 10 is 8.

Solution:
The connection between the centers is also the connection between the nearest points on the sphere.

5) **T** F   If two vectors $\vec{v}$ and $\vec{w}$ are both parallel and perpendicular, then one of the vectors must be the zero vector.

Solution:
If $\vec{v} = \lambda \vec{w}$, then $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| = 0$ implies that one must be empty.

6) T **F**   The curvature $\kappa(\vec{r}(t))$ is always smaller or equal than the length $|\vec{r}\,''(t)|$ of the acceleration vector $\vec{r}\,''(t)$.

Solution:
If you drive along a circle very slowly the acceleration is small but the curvature is the same.

7)   **T**   F   The curve $\vec{r}(t) = \langle \cos(t)\sin(t), \sin(t)\sin(t), \cos(t) \rangle$ is located on a sphere.

Solution:
Check $x^2 + y^2 + z^2 = 1$.

8)   **T**   F   The surface $x^2 + y^2 + z^2 = 2z$ is a sphere.

Solution:
Complete the square to see that it is indeed a sphere centered at $(0, 0, 1)$ with radius 1.

9)   **T**   F   The length of the vector $\langle 4, 2, 4 \rangle$ is an integer.

Solution:
It is the square root of $4^2 + 2^2 + 4^2$ which is 6.

10)   T   **F**   The curvature of the curve $\langle 2\cos(t^3), 2\sin(t^3), 1 \rangle$ is constant 2.

Solution:
It is 1/2.

11)   T   **F**   The graph of the function $f(x, y) = x^2 - y^2$ is called an elliptic paraboloid.

Solution:
It is a hyperbolic paraboloid.

12) T **F**   The equation $\phi = 3\pi/2$ in spherical coordinates defines a plane.

Solution:
$\phi = 3\pi/2$ is out of range

13) **T** F   The vector $\langle 1, 2, 3 \rangle$ is perpendicular to the plane $2x + 4y + 6z = 4$.

Solution:
The vector $\langle 2, 4, 6 \rangle$ is the normal vector.

14) T **F**   The cross product between $\langle 2, 3, 1 \rangle$ and $\langle 1, 1, 1 \rangle$ is 6.

Solution:
It is the dot product which is 6. The cross product is a vector.

15) **T** F   The curve $\vec{r}(t) = \langle \cos(t), t^2, \sin(t) \rangle, 1 \leq t \leq 9$ and the curve $\vec{r}(t) = \langle \cos(t^2), t^4, \sin(t^2) \rangle, 1 \leq t \leq 3$ have the same length.

Solution:
This is a change of parametrization

16) T **F**   If a stone falls for 3 seconds from height $z = h$ to the ground $z = 0$ with gravitational acceleration -10 then the height is 30 meters.

Solution:
The time and height is related by $5t^2 = h$ so that the height is about 45 meters.

17) T **F**   The point $(1, -1, 1)$ has the spherical coordinates the form $(\rho, \theta, \phi) = (\sqrt{3}, \pi/4, \pi/4)$.

Solution:
Apply the transformation formulas. We have $\theta = -pi/4$.

18) | T | **F** | The point $(1, -1, 1)$ has the cylindrical coordinates the form $(r, \theta, z) = (\sqrt{3}, \pi/4, 1)$.

Solution:
Apply the transformation formulas. The correct answer is $(\sqrt{2}, -\pi/4, 1)$.

19) | **T** | F | The distance between two parallel lines in space is the distance of a point on one line to the other line.

Solution:
This is only true for parallel lines.

20) | **T** | F | For two nonzero arbitrary vectors $\vec{v}$ and $\vec{w}$ the identity $\text{Proj}_{\vec{v}}(\vec{v} \times \vec{w}) = \vec{0}$ holds.

Solution:
Make a change of variables.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match contour maps with functions $f(x,y)$. Enter O, where no match.
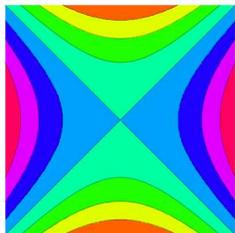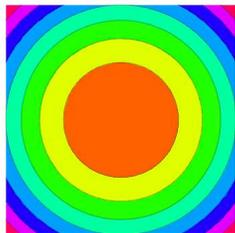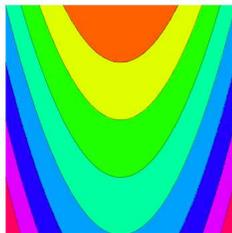
I   II   III

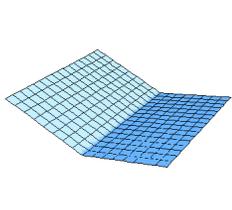| Function $f(x,y) =$	Enter I,II,III
$x^2 + y^2$	
$x^2 - y^2$	
$x^2 - y$	
$x - y^2$	

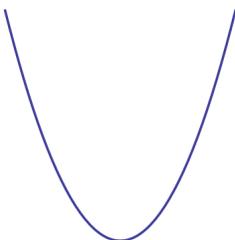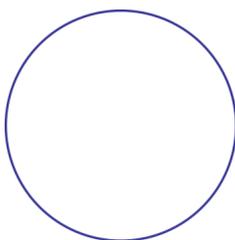b) (3 points) Match the graphs with the functions $f(x,y)$. Enter O, where no match.
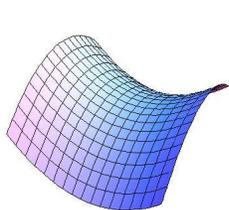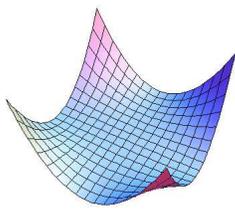
I   II   III

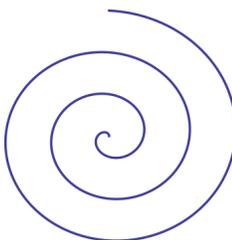| Function $f(x,y) =$	Enter I,II,III
$x - y$	
$|x| - y$	
$x^2 - y^2$	
$x^2 y^2$	
$x - y^3$	

c) (2 points) Match the curves with their parametrizations $\vec{r}(t)$. Enter O, where no match.

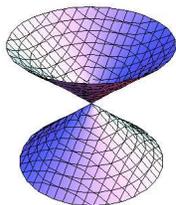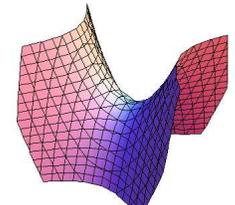

I   II   III

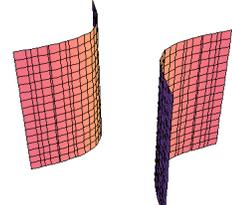| Curve $\vec{r}(t) =$	Enter I,II,III
$\langle t, t^2 \rangle$	
$\langle t^4, 1 + 2t^4 \rangle$	
$\langle -t\sin(t), t\cos(t) \rangle$	
$\langle \sin(t), \cos(t) \rangle$	

d) (3 points) Match level surfaces with definition $g(x,y,z) = 0$. Enter O, where no match.
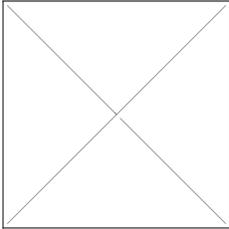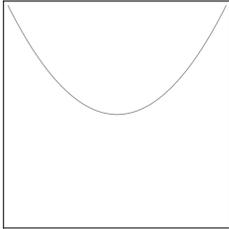

I   II   III

Function $g(x,y,z) =$	Enter I,II,III
$x^2 + y^2 - z^2$	
$x^2 - y^2 - 1$	
$x^2 - y^2 - z$	
$x^2 + y^2 - z$	
$x - y + z$	

Solution:
a) II,I,III,O
b) O,III,I,II,O
c) I,O,III,II
d) I,III,II,O,O

Problem 3) (10 points) No justifications are needed in this problem

3a) (5 points) Matching traces with surfaces.

xy-trace   yz-trace   xz-trace   xy-trace   yz-trace   xz-trace

A   B

C   D

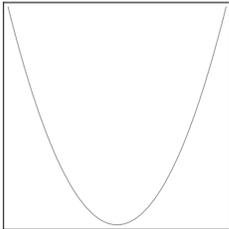The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2 + y^2 - (z - 1/3)^2 = 0$
	$x^2 - y^2 + z = 0$
	$x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + y^2 - z = 1$

Solution:
III,II,I,O
O,III,O,I,II
I,O III,II
O,III,I,II,O

3b) (5 points) Matching parametrized surfaces.
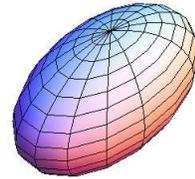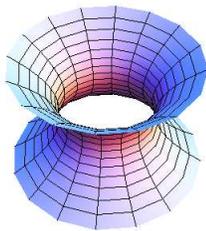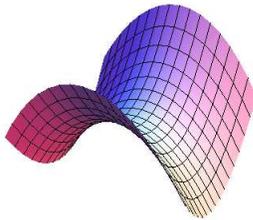
I
II
III
IV

Match the para-
metric surfaces
with their param-
eterization. No
justifications are
needed.

Enter I,II,III,IV here	Parametrization
	$\vec{r}(u,v) = \langle u, v, v^2 - u^2 \rangle$
	$\vec{r}(u,v) = \langle \cos(u)\sin(v), 2\sin(u)\sin(v), \cos(v) \rangle$
	$\vec{r}(u,v) = \langle (v^2+1)\cos(u), (v^2+1)\sin(u), v \rangle$
	$\vec{r}(u,v) = \langle u, 3, v \rangle$

Solution:
a) DABC
b) I,IV,II,III

Problem 4) (10 points)

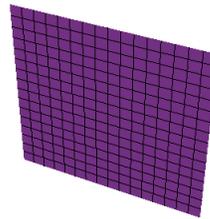We want to find the distance between the lines $x = y = z$ and $(x-1)/2 = (y-2)/3 = (z-4)/4$.

a) (4 points) Find a parametrization for each of the two lines.

b) (6 points) Find the distance between the two lines.

Solution:
a) $\boxed{\vec{r}(t) = \langle t, t, t \rangle}$

$\boxed{\vec{R}(t) = \langle 1 + 2t, 2 + 3t, 4 + 4t \rangle}$.

b) $\vec{n} = \langle 1, 1, 1 \rangle \times \langle 2, 3, 4 \rangle = \langle 1, -2, 1 \rangle$. The connection vector between the first line and second line is $\vec{PQ} = \langle 1, 2, 4 \rangle$. Project it onto $\vec{n}$ to get

$$\frac{|\langle 1, 2, 4 \rangle \cdot \langle 1, -2, 1 \rangle|}{|\langle 1, -2, 1 \rangle|} = \frac{1}{\sqrt{6}} .$$

Problem 5) (10 points)

At the **independence day** celebration on July 4, 2010 in Boston, two rockets were launched at the same time. Their paths follow parabola:

$$\vec{r}(t) = \langle t, t, 5 - t^2 \rangle ,$$

$$\vec{R}(t) = \langle 2 - t, t, 4 + t - t^2 \rangle .$$

a) (3 points) They collide at some time $t = t_0$. Find this time.

b) (4 points) Compute the two velocity vectors $\vec{r}\,'(t)$ and $\vec{R}\,'(t)$ at $t = t_0$.

c) (3 points) Determine the cos of the angle of intersection between the curves at the impact point.

Solution:
a) The time is $\boxed{t=1}$ as can be seen by solving $\vec{r}(t) = \vec{R}(t)$. (Just look at the first component for example.)
b) The velocity vectors are

$$\vec{r}\,'(t) = \langle 1,1,-2t \rangle ,$$
$$\vec{R}\,'(t) = \langle -1,1,1-2t \rangle .$$

Therefore,
$\vec{r}\,'(1) = \boxed{\langle 1,1,-2 \rangle}$ and
$\vec{R}\,'(1) = \boxed{\langle -1,1,-1 \rangle}.$

c) The impact angle is the angle between of the velocity vectors at the point $\vec{r}(1) = \vec{R}(1)$. The cos of the angle is

$$\frac{\vec{r}\,'(1) \cdot \vec{R}\,'(1)}{|\vec{r}(1)| \cdot |\vec{R}'(1)|} = 2/(\sqrt{6}\sqrt{3}) ,$$

which simplifies to $\boxed{\sqrt{2}/3}$.

Problem 6) (10 points)

An octahedron has 4 vertices $A = (-1,-1,0), B = (1,-1,0), C = (1,1,0), D = (-1,1,0)$ in the xy plane. Two other vertices are at $E = (0,0,a)$ and $F = (0,0,-a)$.

a) (4 points) For which positive value of a is the distance between A and F equal to 2 and the solid a regular octahedron?

b) (6 points) Find the distance between A and the line connecting the points B and F.
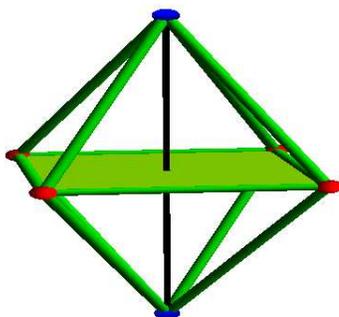
Solution:

a) $2 + a^2 = 4$ shows $a = \sqrt{2}$.

b) $|\langle -1, 1, -a \rangle \times \langle -2, 0, 0 \rangle| / \sqrt{2 + a^2} = \sqrt{12}/2 = \boxed{\sqrt{3}}$.

Problem 7) (10 points)

Let $\vec{v} = \langle 3, 4, 5 \rangle$, $\vec{w} = \langle 1, 1, 1 \rangle$. Compute the following expressions:

a) (2 points) the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$,

b) (2 points) the vector $(\vec{v} \times \vec{w}) \times \vec{w}$,

c) (2 points) the scalar $\vec{v} \cdot \vec{w}$,

d) (2 points) the vector $\mathrm{Proj}_{\vec{v}}(\vec{w})$,

e) (2 points) $\cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$.

Solution:

a) $|\vec{v} \times \vec{w}| = \boxed{\sqrt{6}}$.

b) $\vec{v} \times \vec{w} = \boxed{\langle -1, 2, -1 \rangle}$.

$(\vec{v} \times \vec{w}) \times \vec{w} = \boxed{\langle 3, 0, -3 \rangle}$.

c) $\vec{v} \times \vec{w} = \boxed{12}$

d) $\mathrm{Proj}_{\vec{v}}(\vec{w}) = \vec{v} \cdot \vec{w}/|\vec{w}|^2 \vec{v} = \boxed{\frac{12}{50} \cdot \langle 3, 4, 5 \rangle}$.

e) $\cos(\alpha) = 12/\sqrt{150} = \boxed{2\sqrt{6}/5}$.

Problem 8) (10 points)

a) (7 points) Find the arc length of the curve

$$\vec{r}(t) = \langle \cos(t^2), \sin(t^2), t^2 \rangle$$

from $0 \le t \le 3$.

b) (3 points) Find the unit tangent vector $\vec{T}(t)$ to $\vec{r}(t)$ at time $t = \sqrt{\pi/2}$.

Solution:

a) The velocity is $\vec{r}'(t) = \langle -2t\sin(t^2), 2t\cos(t^2), 2t \rangle$ which has length $2\sqrt{2}t$. This is the speed. Integrating this from 0 to 3 gives $\boxed{9\sqrt{2}}$.

b) The speed is $2\sqrt{2}t$ so that $\vec{T}(t) = \langle -\sin(t^2), \cos(t^2), 1 \rangle / \sqrt{2}$. At $t = \sqrt{\pi/2}$ we have $\boxed{\langle -1, 0, 1 \rangle / \sqrt{2}}$.

Problem 9) (10 points)

A stunt man wants to jump from the golden gate bridge from 80 meters starting at $\vec{r}(0) = \langle 0, 0, 80 \rangle$. He moves from the platform with initial velocity $\vec{r}'(0) = \langle 0, 1, 0 \rangle$. By the way, the swiss **Oliver Favre** holds the record of jumping from 54 meters.

a) (2 points) How long does the diver fall, if the acceleration is $\langle 0, 0, -10 \rangle$.

b) (3 points) Find the trajectory $\vec{r}(t)$ of the stunt man.

c) (3 points) At which point does he hit the water surface $z = 0$?

d) (2 points) With what speed does he hit the surface?

Solution:

a) Look at the height at time t. One can do b) first here. The equation $80 = 5t^2$ has the solution $\boxed{t = 4}$.

b) Integrate
$$\vec{r}''(t) = \langle 0, 0, -10 \rangle$$
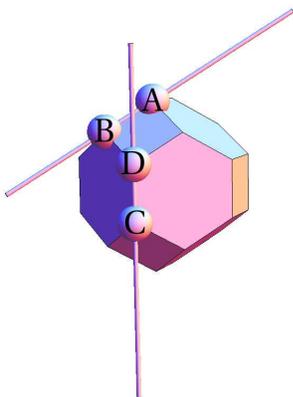twice. After one integration, we get the velocity
$$\vec{r}'(t) = \langle 0, 0, -10t \rangle + \langle 0, 1, 0 \rangle \, .$$

Integrating this again, gives $\boxed{\vec{r}(t) = \langle 0, t, 80 - 5t^2 \rangle}$.

c) Especially, $\vec{r}(4) = \langle 0, 4, 0 \rangle$.

d) We compute the velocity at the impact point: $\vec{r}'(t) = \langle 0, 1, -10t \rangle$ which is $\vec{r}'(4) = \langle 0, 1, -40 \rangle$ has length $\sqrt{1601}$. With 40 meter per second, we have $144km/h$ which is about 90 miles per hour.

A **truncated octahedron** has an edge connecting the vertices $A = (-1, 3, 0), B = (-1, 1, -1)$ and an edge connecting the vertices $C = (-3, -1, 0), D = (-3, 1, 0)$.

a) (5 points) Find the distance of C to the line through A, B.

b) (5 points) Find the distance between the line L through A, B and the line K through C, D.
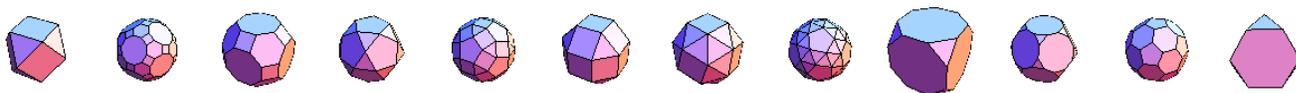
Solution:

a) These are typical distance formula problems: the first is a distance between a point and a line

$$d = \frac{|\vec{AC} \times \vec{AB}|}{|\vec{AB}|} = 6/\sqrt{5} \ .$$

b) The second problem asks for the distance between a line and a point:

$$d = \frac{|\vec{AC} \cdot (\vec{AB} \times \vec{CD})|}{|\vec{AB} \times \vec{CD}|} = |-4|/2 = 2 \ .$$

Here are the remaining 12 **Archimedean solids**. These are polyhedra bound by different types of regular polygons but for which each vertex of the polyhedron looks the same. There are 13 such semiregular polyhedra. Archimedes studied them first in 287BC. Kepler was the first to describe the complete set of 13 in his work **"Harmonices Mundi"** of 1619.