8/6/2015 PRACTICE EXAM II **Maths 21a, O. Knill, Summer 2015**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

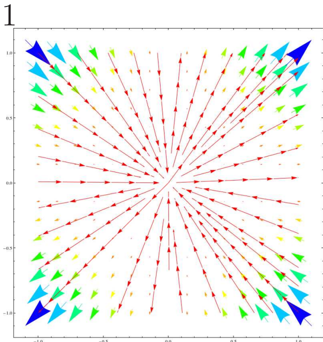
1)	T	F	The equation $f_{xx} = f_y$ is the heat equation, an example of an partial differential equation.
2)	T	F	The triple scalar product between three vectors in a plane is always zero.
3)	T	F	The length of the gradient $|\nabla f|$ is maximal at a maximum of f.
4)	T	F	Given a line L inside a plane Σ and a point P. The distance between L and P is larger or equal than the distance from Σ and P.
5)	T	F	For a curve on the sphere, the $\vec{B}$ vector in the TNB frame points always outwards.
6)	T	F	The surface $x^2 - y^2 + z^2 = -1$ is a one sheeted hyperboloid.
7)	T	F	The equation $u_{tt} = uu_x + u_{xx}$ is called the Burger's equation.
8)	T	F	The fundamental theorem of line integrals assures that the line integral of any gradient field along any path zero.
9)	T	F	The dot product between $\vec{v} = \langle 1, 1 \rangle$ and $\vec{w} = \langle 1, 0 \rangle$ is is equal to the cross product $\vec{v} \times \vec{w}$ between these two vectors.
10)	T	F	If $\vec{T}(t)$ is the unit tangent vector. Then $\vec{T}' = \vec{0}$ because the length of $\vec{T}$ stays the same.
11)	T	F	We can look at the discriminant $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ at every point (x, y) If D has a maximum at $(0, 0)$, then the gradient of D is zero at $(0, 0)$.
12)	T	F	The curl of a conservative field is zero. [PS: this is the reason why all republicans have straight hair!]
13)	T	F	The flux of the curl of $\vec{F}$ through the surface S is positive, then S has a boundary.
14)	T	F	$\vec{r}(u, v) = \langle u^3, 0, v^3 \rangle$ parametrizes a plane.
15)	T	F	If the integral $\int\int\int_G \text{div}(\vec{F}(x, y, z)) \, dxdydz$ is zero for all balls $G = \{x^2 + y^2 + z^2 \leq r\}$, then the divergence is zero.
16)	T	F	Assume $\vec{r}(t)$ is a stream line of a vector field $F = \nabla f$. Then the velocity of $\vec{r}$ is zero if we are at a critical point of f.
17)	T	F	If $\vec{F} = \text{curl}(\vec{G})$ and $\text{div}(\vec{F}) = 0$, then $\text{div}(\vec{G}) = 0$.
18)	T	F	It is possible that $\vec{v} \cdot \vec{w} > 0$ and $\vec{v} \times \vec{w} = \vec{0}$.
19)	T	F	The directional derivative $D_{\vec{v}}(r)$ of a curve $\vec{r}(t)$ is defined as $\vec{v} \cdot \vec{r}\,'(t)$.
20)	T	F	The vector projection $P_{\vec{w}}(\vec{v})$ defines a product which is commutative: $P_{\vec{w}}(\vec{v}) = P_{\vec{v}}(\vec{w})$.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the following vector fields

Field	Enter 1-4
$\vec{F}(x,y) = \langle x^2y, y^2x \rangle$	
$\vec{F}(x,y) = \langle x^2y, -y^2x \rangle$	
$\vec{F}(x,y) = \langle xy^2, x^2y \rangle$	
$\vec{F}(x,y) = \langle xy^2, -x^2y \rangle$	

1



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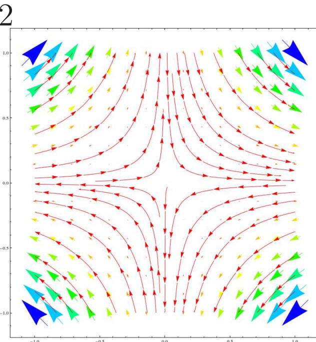

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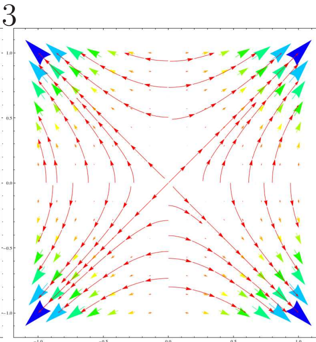

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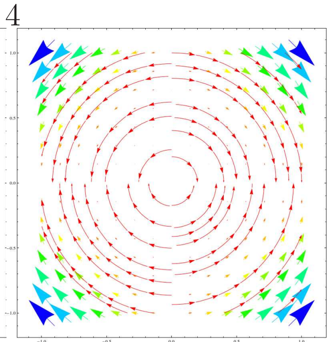


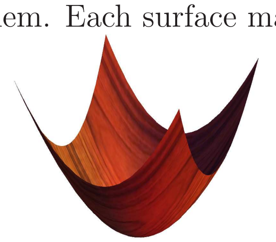
b) (2 points) Match the following names of partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
Laplace		
heat		
wave		
transport		

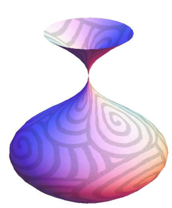
A	$u(t,x) = t^2 + x^2$
B	$u(t,x) = t^2 - x^2$
C	$u(t,x) = \sin(x+t)$
D	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_{xx}(t,x)$
2	$u_t(t,x) = u_x(t,x)$
3	$u_{tt}(t,x) = -u_{xx}(t,x)$
4	$u_{tt}(t,x) = u_{xx}(t,x)$

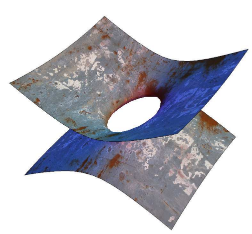
c) (4 points) The following surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.



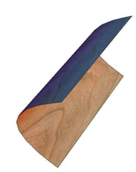
A



B



C



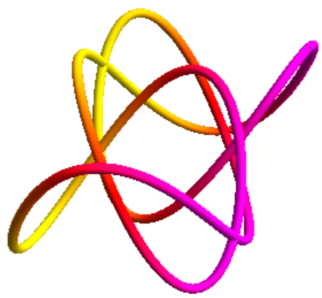
D

Enter A-D here	Function or parametrization
	$x - 4z^2 = 1$
	$\vec{r}(u,v) = \langle (1+\sin(u))\cos(v), (1+\sin(u))\sin(v), u \rangle$
	$\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$
	$4x^2 + y^2 - 9z^2 = 1$

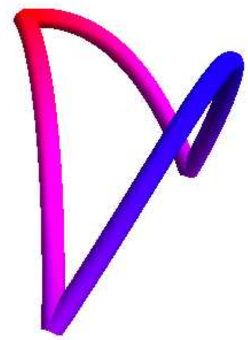
Problem 3) (10 points) No justifications are necessary

a) (3 points) Closed curves in space are called knots. Match them

Formula	Enter A,B,C,D
$\vec{r}(t) = \langle \lvert\cos(t)\rvert, \lvert\sin(t)+\cos(t)\rvert, \lvert\cos(2t)\rvert \rangle$	
$\vec{r}(t) = \langle \cos(3t), \cos(t)+\sin(3t), \cos(5t) \rangle$	
$\vec{r}(t) = \langle (2+\cos(\frac{3t}{2}))\cos(t), (2+\cos(\frac{3t}{2}))\sin(t), \sin(\frac{3t}{2}) \rangle$	
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$	



A



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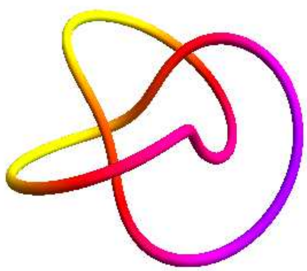

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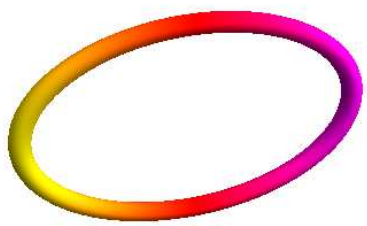


D

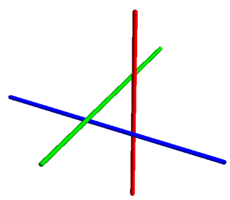
b) (5 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

Theorem	Grad	Curl	Div	Line integral	Flux integral
Fundamental Thm line integrals					
Greens theorem					
Divergence theorem					
Stokes' theorem					
Second derivative test					

c) (2 points) Which of the following formulas give the unit tangent vector?

label	formula
A	$\int_a^b \lvert\vec{r}\,'(t)\rvert \, dt$
B	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
C	$\vec{T}\,'(t)/\lvert\vec{r}\,'(t)\rvert^3$
D	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

Problem 4) (10 points)



a) (4 points) Find a parametrization $\vec{r}(t) = P + t\vec{v}$ of the line defined by $x = 1, y = 1$.
b) (6 points) Find all the three distances between the line

$$x = 1, y = 1 ,$$

the line

$$x = -1, z = 1$$

and the line

$$y = -1, z = -1 .$$

Problem 5) (10 points)

Picture credit: (http://chibionpu.webs.com)

Estimate

$$\left(\frac{10001 \cdot 9999}{10002}\right)^{1/4}$$

Magic!

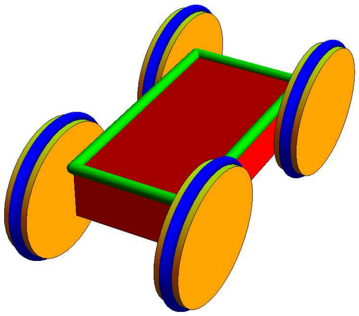
Problem 6) (10 points)

Find and classify the minima and maxima of the function

$$f(x, y) = 2x^3 - 15x^2 + 36x - y^2 - 2y .$$

The function is remarkable and special because it does not have any interpretation whatsoever. It is therefore a "very special function" and dear to us. Determine also whether f has a global maximum somewhere.

Problem 7) (10 points)



We build a toy car. Minimize the weight

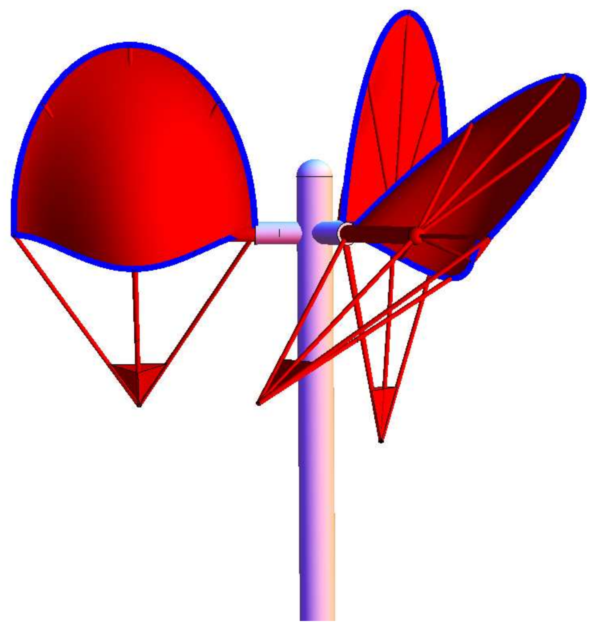
$$f(x,y) = x^2 + 4\pi y^2$$

of the car with length=width x and wheel radius y under the constraint that the combined steel frame length and tire length

$$g(x,y) = 4x + 4\pi y = 8$$

is constant.

Problem 8) (10 points)



Porter square in Cambridge features a moving art project. It proves that paddle wheels and curl are omni present. To build that model for google earth, we idealized one of the blades as the surface

$$\vec{r}(t,s) = \langle 2\cos(t)\sin(s), 4\sin(t)\sin(s), \cos^4(s)\rangle$$

with $t, s \in [0, \pi]$. The surface S is oriented so that the boundary consists of two parts. The surface boundary can be parametrized as

$$\vec{r}_1(t) = \langle 2\sin(t), 0, \cos^4(t)\rangle, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

and

$$\vec{r}_2(t) = \langle 2\cos(t), 4\sin(t), 0\rangle, 0 \leq t \leq \pi \ .$$

The wind force is given by the curl of the vector field $\vec{F}(x,y,z) = \langle y, z, 0\rangle$. Find the flux of $\text{curl}(\vec{F})$ through the surface S.

Problem 9) (10 points)

Image: Pforzheim Jewelry Museum.

August Kiehnle, 1885

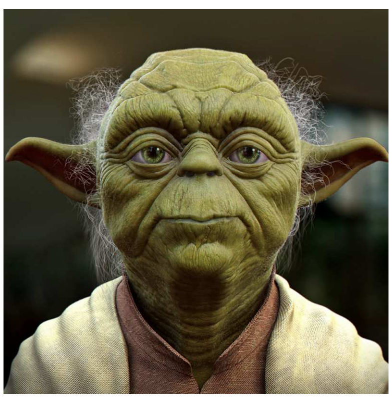
A chain has the shape:

$$\vec{r}(t) = \langle 2t, 18 + e^t + e^{-t}, 0 \rangle \ .$$

a) (6 points) What is the length of this chain parametrized for $-1 \leq t \leq 1$.

b) (4 points) You are so excited, you drop the chain. The center piece which is initially at $\vec{r}(0) = \langle 0, 0, 20 \rangle$ will fall on the z axes while subject to the gravitational force $\vec{r}\,''(0) = \langle 0, 0, -10 \rangle$ and having zero velocity $\vec{r}\,'(0) = \langle 0, 0, 0 \rangle$ initially. How long does it take to hit the ground?

Problem 10) (10 points)

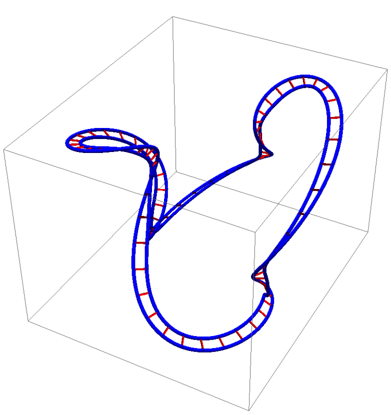


Evaluate the following integral

$$\int_0^4 \int_{\sqrt{y/4}}^1 \int_0^{e^{-x^4}} x \, dz \, dx dy \ .$$

Remember what Yoda said.

Problem 11) (10 points)

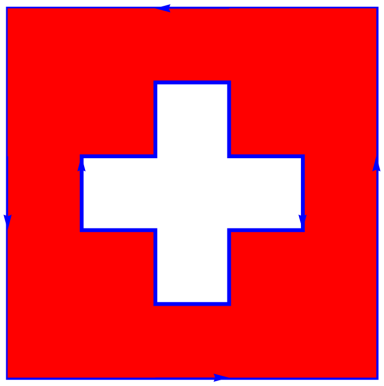


In 2006, Oliver submitted a roller coaster suggestion to a Harvard sustainability competition. Imagine to ride from the science center to Allston in a minute with some loop over the Charles. Together with a coffee shop on top of the science center and hot dog and coffee stands near the river and Storrow drive under ground, Cambridge would become the funnest city in the world. So far, it is only a dream. Assume the roller coaster is the path

$$\vec{r}(t) = \langle 4\cos(t) + \sin(5t), 4\sin(t) - 2\cos(5t), 4\sin(2t) \rangle$$

and assume, the car experiences a force $\vec{F}(x, y, z) = \langle x^{100}, y, z^{20} \rangle$. What work is done to go from $\vec{r}(0)$ to $\vec{r}(\pi)$?

Problem 12) (10 points)



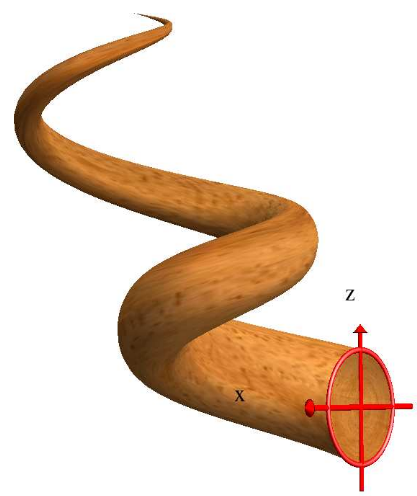
Let G be the red complement of the cross in the Swiss flag. The entire flag has dimension 5×5 and the cross consists of 5 squares of unit length. Let C be the boundary of the red complement region oriented so that the region is to the left. The boundary consists of two curves. Find the line integral

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) ,$$

where

$$\vec{F}(x,y) = \langle x^9 y^{10} + 7y, y^9 x^{10} - 3x + \sin(y) \rangle .$$

Problem 13) (10 points)



A twisted Swiss Alphorn S is parametrized as

$$\vec{r}(u,v) = \langle 4\pi(4 + (2 - \frac{u}{2\pi})\cos(v))\cos(u) - u, 4\pi(4 - \frac{u}{2\pi}\cos(v))\sin(u), 4\pi(2 - \frac{u}{2\pi})\sin(v) + u \rangle ,$$

where $0 \leq u \leq 4\pi$ and $0 \leq v \leq 2\pi$. Together with a disc D obtained in the xz-plane $y = 0$, the closed surface $S \cup D$ bounds the solid E, oriented outwards. You are given that the volume of E is the same than the volume of a cone $x^2 + y^2 \leq (4\pi - z)^2, 0 \leq z \leq 4\pi$. The vector field

$$\vec{F}(x,y,z) = \langle -y + x, x + z^2, z + x^2 \rangle$$

is the sound energy transfer field. Find the flux of $\vec{F}$ through $S \cup D$.