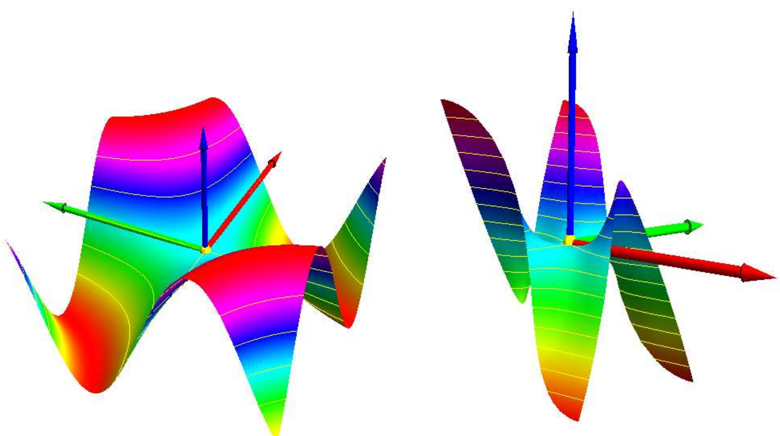
Lecture 5: Functions

A **function of two variables** $f(x, y)$ is a rule which assigns to two numbers x, y a third number $f(x, y)$. For example, the function $f(x, y) = x^3y + 2x$ assigns to $(2, 3)$ the number $2^3 \cdot 3 + 4 = 28$.

A function is usually defined for all points (x, y) in the plane like in the case $f(x, y) = x^2 + \sin(xy)$. Sometimes, it is required to restrict the function to a **domain** D in the plane. For example, if $f(x, y) = \log|y| + \sqrt{x}$, then (x, y) is only defined for $x > 0$ and $y \neq 0$. The **range** of a function f is the set of values which the function f can take. The function $f(x, y) = 3 + x^2/(1 + x^2)$ for example takes all values $3 \leq z < 4$ and the value $z = 4$ is not attained.

The **graph** of $f(x, y)$ is the set $\{(x, y, f(x, y)) \mid (x, y) \in D\}$ in $\mathbb{R}^3$. Graphs are surfaces which allow to see the function visually.



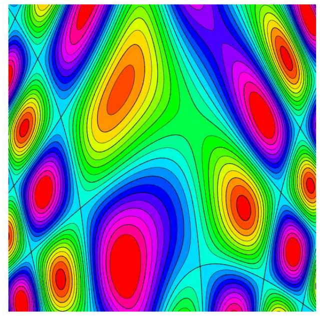
1 The graph of $f(x, y) = \sqrt{1 - x^2 - y^2}$ on the domain $x^2 + y^2 < 1$ is a half sphere.

2

Example $f(x, y)$	domain D of f	range = $f(D)$ of f
$-\log(1 - x^2 - y^2)$	open unit disc $x^2 + y^2 < 1$	$(0, \infty)$
$f(x, y) = x^2 + y^3 - xy + \cos(xy)$	plane R^2	the real line
$\sqrt{4 - x^2 - 2y^2}$	$x^2 + 2y^2 \leq 4$	$[0, 2]$
$1/(x^2 + y^2 - 1)$	all except unit circle	$R \setminus (1, 0]$
$1/(x^2 + y^2)^2$	all except origin	positive real axis

The set $\{(x, y) \mid f(x, y) = c = \text{const}\}$ is called a **contour curve** or **level curve** of f. For example, for $f(x, y) = 4x^2 + 3y^2$, the level curves $f = c$ are ellipses if $c > 0$. Drawing several contour curves $\{f(x, y) = c\}$ simultaneously produces a **contour map** of the function f.

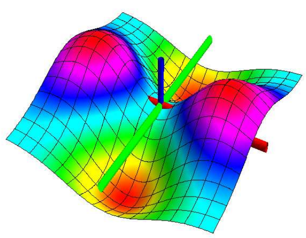
Level curves allow to visualize functions of two variables $f(x,y)$ without leaving the plane. The picture to the right for example shows the level curves of the function $\sin(xy) - \sin(x^2 + v)$. Contour curves are encountered every day: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

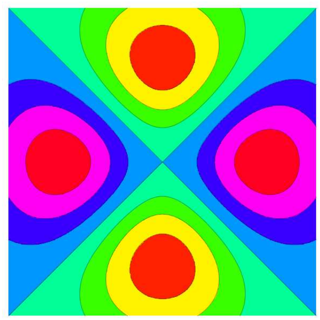


3 For $f(x,y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

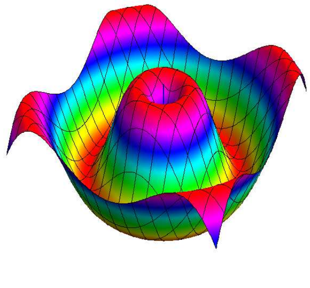
4 The function $f(x,y) = 1 - 2x^2 - y^2$ has contour curves $f(x,y) = 1 - 2x^2 + y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.

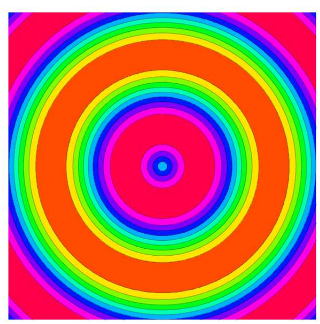
5 For the function $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2 - y^2)e^{-x^2-y^2} = c$. We can draw the curves however with the help of a computer:





6 The surface $z = f(x,y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.





In applications, discontinuous functions can occur. The temperature of water in relation to pressure and volume for example, one experiences **phase transitions**, places where the function value can jump. Mathematicians have tamed singularities in a mathematical field called "catastrophe theory".

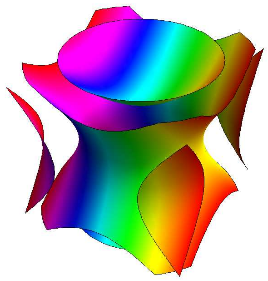
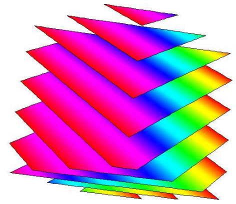
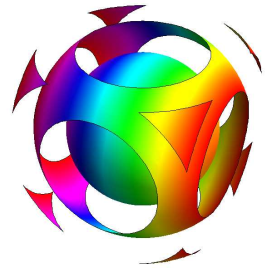
A function $f(x,y)$ is called **continuous** at (a,b) if there is a finite value $f(a,b)$ with $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n, y_n) converging to (a,b), also $f(x_n,y_n) \to f(a,b)$. A function is **continuous** in a subset G of the plane if it is continuous at every point (a,b) of G.

Continuity means that if (x,y) is close to (a,b), then $f(x,y)$ is close to $f(a,b)$. Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check. Fortunately however, we do not have to worry about it most of the time. Lets look at some examples:

7 **Example:** For $f(x,y) = (xy)/(x^2+y^2)$, we have $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous at $(0,0)$.

8 For $f(x,y) = (x^2y)/(x^2+y^2)$, it is better to describe the function using polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous if we extend it and postulate $f(0,0) = 0$.
P.S. It is custom in modern mathematics to consider the function in this example **to be continuous**. The reason is that there is a **canonical way** to give a function value at the undefined point. This happens at other places. Mathematics can deal with interesting and powerful ideas: the sum $1+2+4+8+16+...$ does not make any sense at first. But its natural value is -1: the reason is that $1+a+a^2+a^3+... = 1/(1-a)$ extends the meaning of the left hand side if a is set to 2. The extended definition to the right gives the value -1 even so the sum definition does not make sense any more. Such ideas are important in fundamental questions and pivotal to understand the **secrets about prime numbers**. The famous **Riemann hypothesis** for example claims that all the zeros of the function $f(s) = 1+1/2^s+1/3^s+...$ are located on the axes, where the real part of s is $1/2$. Also here, the sum itself does not make sense for values like $s = 1/2+i$ but we know how to interpret it in a canonical way, and even evaluate it for $s = 0$, where it has the value $-1/2$. Back to the example: the function $f(x,y) = (x^2y)/(x^2+y^2)$ contains already all the information we need to define it at $(x,y) = 0$ even so the formula itself can not be evaluated at $(x,y) = (0,0)$. Some calculus books would insist that the function is not continuous. We are more generous and consider it to be continuous. Similarly, we would immediately say that functions like $f(x,y) = y(x^2-1)/(x+1)$ or $f(x,y) = \sin(x)y^2/x$ are continuous everywhere. Indeed in both cases, there is a natural and unique continuous and even analytic continuation. What we have just addressed is a change in the perception of the function concept which happened pretty late, maybe with Fourier and then with Riemann. Before Fourier, one always insisted on a function to have an analytic expression. Before Riemann, one would not look at the world hidden behind analytic continuation.

A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. The function $f(x,y,z) = x^2+y-z$ for example satisfies $f(3,2,1) = 10$. We can visualize a function by **contour surfaces** $g(x,y,z) = c$, where c is constant. It is an **implicit description** of the surface. The contour surface of $g(x,y,z) = x^2+y^2+z^2 = c$ for example is a sphere if $c > 0$. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

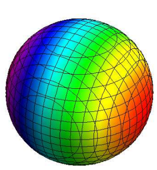
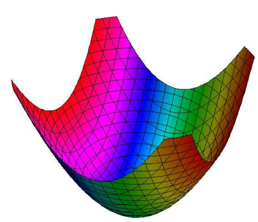
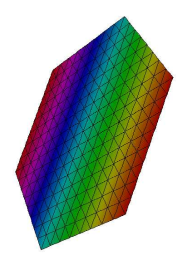
9 The function $g(x, y, z) = 2 + \sin(xyz)$ could define a temperature distribution in space. We can no more draw the graph of g because that would be an object in 4 dimensions. We can however draw surfaces like $g(x, y, z) = 0$.

10 The level surfaces of $g(x, y, z) = x^2 + y^2 + z^2$ are spheres. The level surfaces of $g(x, y, z) = 2x^2 + y^2 + 3z^2$ are ellipsoids.

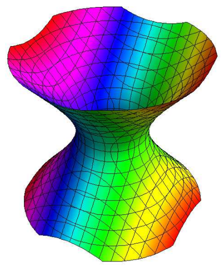
11 For $g(x, y, z) = z - f(x, y)$, the level surface $g = 0$ agrees with the graph $z = f(x, y)$ of f. For example, for $g(x, y, z) = z - x^2 - y^2 = 0$, the graph $z = x^2 + y^2$ of the function $f(x, y) = x^2 + y^2$ is a paraboloid. Graphs are special surfaces. Most surfaces of the form $g(x, y, z) = c$ can not be written as graphs. The sphere is an example. We would need two graphs to cover it.

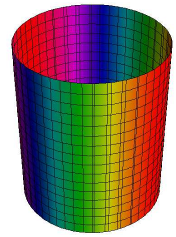
12 The equation $ax + by + cz = d$ is a plane. With $\vec{n} = \langle a, b, c\rangle$ and $\vec{x} = \langle x, y, z\rangle$, we can rewrite the equation $\vec{n} \cdot \vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n} \cdot \vec{x}_0 = d$. so that $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$. This means that every vector $\vec{x} - \vec{x}_0$ in the plane is orthogonal to $\vec{n}$.

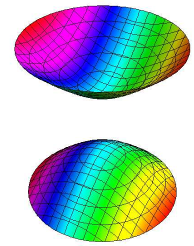
13 For $f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m$ the surface $f(x, y, z) = 0$ is called a **quadric**. We will look at a few examples.

Sphere	Paraboloid	Plane
		
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$	$(x-a)^2 + (y-b)^2 - c = z$	$ax + by + cz = d$

One sheeted Hyperboloid	Cylinder	Two sheeted Hyperboloid





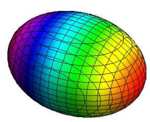


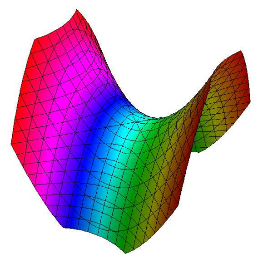
$(x-a)^2+(y-b)^2-(z-c)^2 = r^2$ $(x-a)^2+(y-b)^2 = r^2$ $(x-a)^2+(y-b)^2-(z-c)^2 = -r^2$

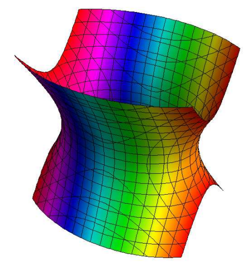
Ellipsoid

Hyperbolic paraboloid

Elliptic hyperboloid

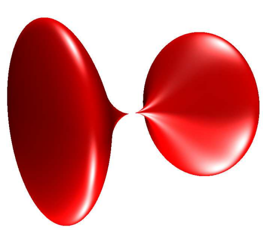


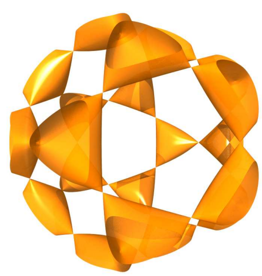


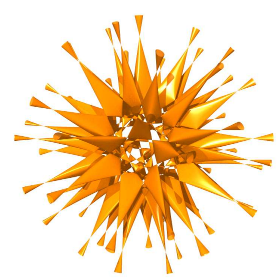


$x^2/a^2+y^2/b^2+z^2/c^2 = 1$ $x^2-y^2+z = 1$ $x^2/a^2+y^2/b^2-z^2/c^2 = 1$

14 Higher order polynomial surfaces can be intriguingly beautiful and are sometimes difficult to describe. If f is a polynomial in several variables and $f(x,x,x)$ is a polynomial of degree d, then f is called a **degree d polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.







Homework

1 Plot the graph of the function $f(x, y) = \sin(x^2)\sin(y^2)$ on the region $0 \leq x \leq \pi, 0 \leq y \leq \pi$. How many mountain peaks do you count inside that region? How many minima do you see? You can do it by hand, or Wolfram alpha or Mathematica or a graphing calculator to check your picture. You do not have to compute the extrema analytically. We will do that later. This is simply a graphing problem.

2 a) Determine the domain and range of the **logarithmic mean** $f(x, y) = \frac{(y-x)}{\log(y)-\log(x)}$, where log the natural logarithm.
b) The function is not defined at $x = y$ but one can define $f(x, y)$ on the diagonal $x = y$. Use Hôpital to show that the limit $\lim_{x\to 2} f(x, 2)$ exists.
c) The function is also not defined at first if $x = 0$ or $y = 0$. Show that the limit $\lim_{x\to 0} f(x, 2)$ exists.

3 a) Use the computer to draw the level surface $x^2 + y^2 + z^2 + x^2y^2z^2 = 20$.
b) Do the same for the $((x^2 + y^2)^2 - x^2 + y^2)^2 + z^2 = 0.01$.

4 a) Sketch the graph and contour map of $f(x, y) = \cos(1 + 2x^2 + y^2)/(1 + 2x^2 + y^2)$.
b) Sketch the contour map of $g(x, y) = 2|x| - 5|y|$.
c) Sketch the contour map of $h(x, y) = (x^2 + y^2)^2 - x^2 + y^2$.

5 a) Verify that the line $\vec{r}(t) = \langle 1, 3, 2\rangle + t\langle 1, 2, 1\rangle$ is part of the hyperbolic paraboloid $z^2 - x^2 - y = 0$.
b) Verify that the line $\vec{r}(t) = \langle 1 + t, 1 - t, t\rangle$ is part of the one sheeted hyperboloid $x^2 + y^2 - 2z^2 = 2$.
c) As also the line $\vec{r}(s) = \langle 1 - s, 1 + s, s\rangle$ is part of the same hyperboloid, what is the intersection of the hyperboloid with the plane $\vec{r}(t, s) = \langle 1 + t - s, 1 - t + s, t + s\rangle$?

Lecture 6: Parametrized surfaces

We have seen surfaces described as level surfaces $g(x, y, z) = c$. A fundamentally different way to describe a surface is giving a **parametrization**. This generalizes the parametrization of a plane given last week.

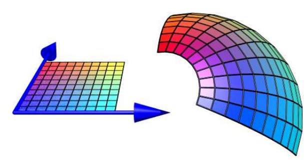
A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle ,$$

where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables. The variables u, v are called **parameters**, defining a coordinate system on the surface.

Because two parameters u and v are involved, the map $\vec{r}$ is also called *uv*-**map**. And like *uv*-light, it looks cool.

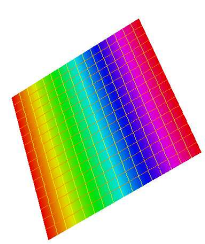
A **parametrized surface** is the image of the uv-map. The domain of the uv-map is called the **parameter domain**. The parametrization is something you are **doing**, the surface itself is something you **see**.



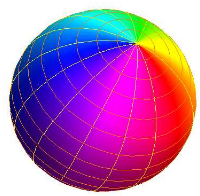
If we keep the first parameter u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called **grid curves**.

A computer draws parametrized surfaces using grid curves. The world of parametric surfaces is intriguing and complex. You can be explore this world with the help of a computer. The parametrization topic is not so easy. But you can survive it by knowing four important examples. They are *really important* because they are cases we can understand well and which consequently will return again and again. They are also building blocks for more general surfaces.

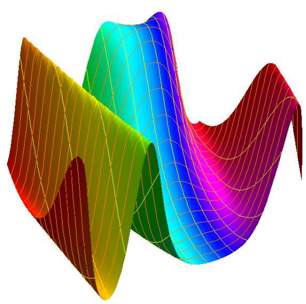
1 **Planes:** Parametric: $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$
Implicit: $ax + by + cz = d$. Parametric to implicit: find the normal vector $\vec{n} = \vec{v} \times \vec{w}$. Implicit to Parametric: find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.



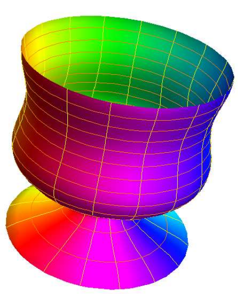
2 **Spheres:** Parametric: $\vec{r}(u,v) = \langle a,b,c \rangle + \langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v) \rangle$.
Implicit: $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.
Parametric to implicit: reading off the radius.
Implicit to parametric: find the center (a,b,c) and the radius r possibly by completing the square.



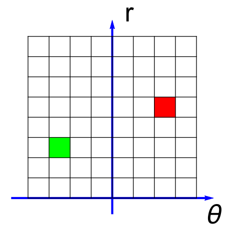
3 **Graphs:**
Parametric: $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$
Implicit: $z - f(x,y) = 0$. Parametric to implicit: look up the function $f(x,y)$ $z = f(x,y)$
Implicit to parametric: use x and y as variables.

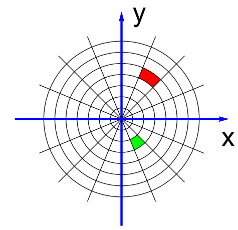


4 **Surfaces of revolution:**
Parametric: $\vec{r}(u,v) = \langle g(v)\cos(u), g(v)\sin(u), v \rangle$
Implicit: $\sqrt{x^2+y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.
Parametric to implicit: read off $g(z)$ the distance to the z-axis.
Implicit to parametric: use the radius function $g(z)$ and think about polar coordinates.



A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have $(x, y) = (r\cos(\theta), r\sin(\theta))$.





The formula $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of an integer multiple of π. The points $(1, 2)$ and $(-1, -2)$ for example have the same θ value. In order to get the correct θ value one can take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the limit when $x \to 0^+$, then add π if $x < 0$ or if $x = 0$ and $y < 0$.

The coordinate system obtained by representing points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

is called the **cylindrical coordinate system**.

Here are some level surfaces in cylindrical coordinates:

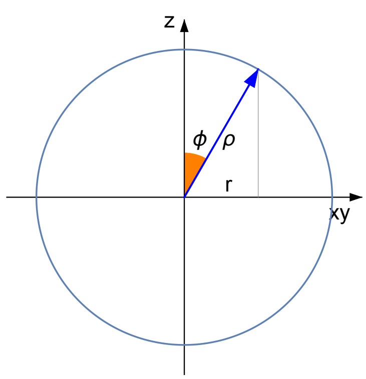
5 $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone**, $r^2 = z$ **elliptic paraboloid**, $\theta = 0$ is a **half plane**, $r = \theta$ is a **rolled sheet of paper**.

6 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

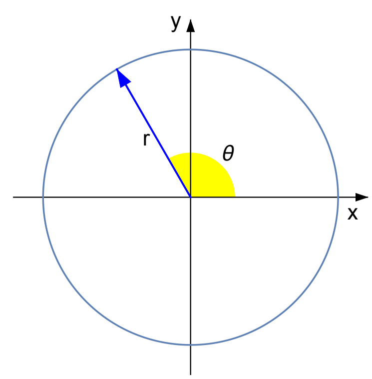
Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ called **Euler angles**. The first angle θ is the angle we have used in polar coordinates. The second angle, ϕ, is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

Here are two important figures. The distance to the z axes $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture.



$$x = \rho\cos(\theta)\sin(\phi),$$
$$y = \rho\sin(\theta)\sin(\phi),$$
$$z = \rho\cos(\phi)$$



Here are some level surfaces described in spherical coordinates:

7 $\rho = 1$ is a **sphere**, the surface $\phi = \pi/4$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

Homework

1 Find a parametrization for the plane which contains the three points $P = (1, 8, 2)$,$Q = (3, 3, 2)$ and $R = (2, 4, 6)$.

2 Plot the surface with the parametrization

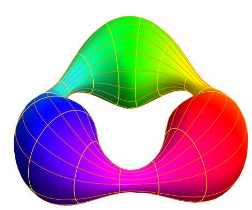
$$\vec{r}(u, v) = \langle \cos(u)\cos(v), \cos(u)\sin(v), (1/(1 + v^2)) \rangle \ ,$$

where $0 \leq u \in 2\pi$ and $-\pi \leq v \leq \pi$. You can use technology if you like.

3 a) Find a parametrisations of the lower half of the ellipsoid $25x^2 + 16y^2 + z^2 = 1, z < 0$ by using that the surface is a graph $z = f(x, y)$.
b) Find a second parametrization but use angles ϕ, θ similarly as for the sphere.

4 Find a parametrisation of the **bumpy torus**, given as the set of points which have distance $3 + 2\cos(3\theta)$ from the circle $\langle 10\cos(\theta), 10\sin(\theta), 0\rangle$, where θ is the angle occurring in cylindrical and spherical coordinates.

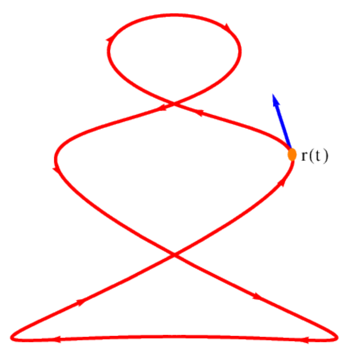
Hint: Use r, the distance of a point $(x, y,)$ to the z-axis. This distance is $r = (10 + (3 + 2\cos(3\theta))\cos(\phi))$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = (3 + 2\cos(3\theta))\sin(\phi)$. To finish the parametrization problem, translate back to Cartesian coordinates.

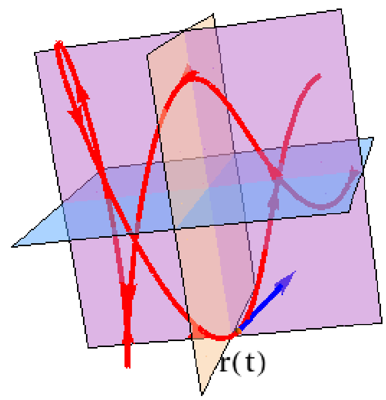


5 a) What is the equation for the surface $x^2 + y^2 = 3x + z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(4\phi)|$ in spherical coordinates (ρ, θ, ϕ).

Lecture 7: Parametrized curves

A **parametrization** of a planar curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t)$ and $y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. The parametrization of a **space curve** is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. The image of $\vec{r}$ is a **parametrized curve** in space.





We always think of the **parameter** t as **time**. Think about the parametrization as what you **do**, while the result, the curve, is what you **see**. For a fixed time t, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains **more information** about the curve then the image, the actual curve alone. It tells for example, how fast we go along the curve.

1 The parametrization $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ is an example of a **Lissajous curve**.

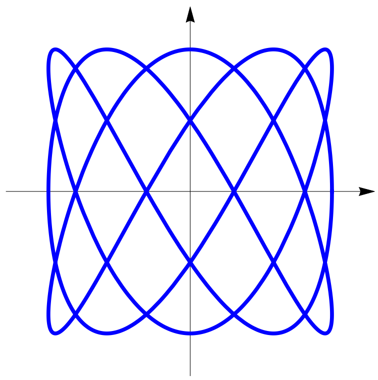
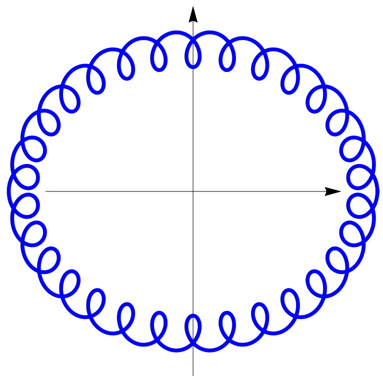
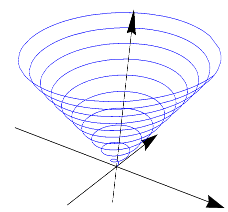
2 If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t) \rangle$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola.

3 With $x(t) = 2\cos(t), y(t) = 3\sin(t)$, then $\vec{r}(t)$ follows an **ellipse**. We can see this from $x(t)^2/4 + y(t)^2/9 = 1$. We can can overlay an other circular motion to get an epicycle $\vec{r}(t) = \langle 2\cos(t) + \cos(31t)/4, 3\sin(t) + \sin(31t)/4 \rangle$.

4 With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$. It traces a **helix** which has a radius changing linearly.

5 If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.

6 If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve again but we traverse it in the **opposite direction**.

7 If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c)\rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q. For example, $\vec{r}(t) = \langle 1 + t, 1 - t, 2 + 3t\rangle$ connects the points $P = (1, 1, 2)$ with $Q = (2, 0, 1)$.

Sometimes it is possible to eliminate the time parameter t and write the curve using equations. An example was the **symmetric equations** for the line. We need one equation to do so in two dimensions but two equations in three dimensions.

If a curve is written as an intersection of two surfaces $f(x, y, z) = 0, g(x, y, z) = 0$, it is called an **implicit description of the curve**.

8 The symmetric equations describing a line $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ is the intersection of two planes.

9 If f and g are polynomials, the set of points satisfying $f(x, y, z) = 0, g(x, y, z) = 0$ is an example of an **algebraic variety**. An example is the set of points in space satisfying $x^2 - y^2 + z^3 = 0, x^5 - y + z^5 + xy = 3$. An other example is the set of points satisfying $x^2 + 4y^2 = 5, x + y + z = 1$ which is an ellipse in space.

10 For $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then $x = t\cos(z), y = t\sin(z)$ and we can see that $x^2 + y^2 = z^2$. The curve is located on a cone. We also have $y/x = \tan(z)$ so that we could see the curve as an intersection of two surfaces. Detecting relations between x, y, z can help to understand the curve.

11 Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time. Here are some more places, where curves appear:

Strings or knots	are closed curves in space.
Molecules	like RNA or proteins.
Graphics:	surfaces are represented by mesh of curves.
Typography:	fonts represented by Bézier curves.
Relativity:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle \dot{x}, \dot{y}, \dot{z}\rangle$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

Any vector parallel to the velocity $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

Here are where velocities, acceleration and jerk are computed:

Position	$\vec{r}\,(t)$	$= \langle \cos(3t), \sin(2t), 2\sin(t)\rangle$
Velocity	$\vec{r}\,'(t)$	$= \langle -3\sin(3t), 2\cos(2t), 2\cos(t)\rangle$
Acceleration	$\vec{r}\,''(t)$	$= \langle -9\cos(3t), -4\sin(2t), -2\sin(t)\rangle$
Jerk	$\vec{r}\,'''(t)$	$= \langle 27\sin(3t), 8\cos(2t), -2\cos(t)\rangle$

Lets look at some examples of velocities and accelerations:

Signals in nerves:	40 m/s	Train:	0.1-0.3 m/s^2
Plane:	70-900 m/s	Car:	3-8 m/s^2
Sound in air:	Mach 1=340 m/s	Free fall:	1G = 9.81 m/s^2
Speed of bullet:	1200-1500 m/s	Space shuttle:	$3G = 30m/s^2$
Earth around the sun:	30'000 m/s	Combat plane F16:	9G m/s^2
Sun around galaxy center:	200'000 m/s	Ejection from F16:	14G m/s^2.
Light in vacuum:	300'000'000 m/s	Electron in vacuum tube:	10^{15} m/s^2

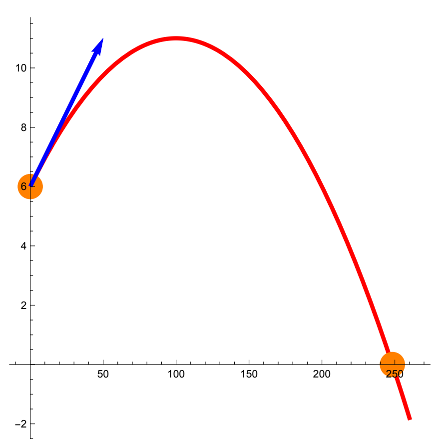
The **addition rule** in one dimension $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.

$(\vec{v}+\vec{w})' = \vec{v}\,' + \vec{w}\,'$, $(c\vec{v})' = c\vec{v}\,'$, $(\vec{v}\cdot\vec{w})' = \vec{v}\,'\cdot\vec{w} + \vec{v}\cdot\vec{w}\,'$ $(\vec{v}\times\vec{w})' = \vec{v}\,'\times\vec{w} + \vec{v}\times\vec{w}\,'$ $(\vec{v}(f(t)))' = \vec{v}\,'(f(t))f'(t)$.

The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\,ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ at all times as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\,ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\,ds$.

The **free fall** is the case when acceleration is constant. The direction of the constant force defines what is "down". If $\vec{r}''(t) = \langle 0,0,-10\rangle$, $\vec{r}'(0) = \langle 0,1000,2\rangle, \vec{r}(0) = \langle 0,0,h\rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2\rangle$.



If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2$.

Homework

1 a) Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle 2 - t^3, t^2 - 6 \rangle$ for $t \in [-1, 1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as the acceleration $\vec{r}\,''(t)$ at $t = 2$.
b) Sketch the space curve
$\vec{r}(t) = \langle (10+\cos(15t))\cos(4t), (10+\cos(15t))\sin(4t), \sin(15t) \rangle$.

2 A device in a car measures the acceleration $\vec{r}\,''(t) = \langle \cos(t), -\cos(9t) \rangle$ at time t. Assume the car is at $(0, 0)$ at time $t = 0$ and has velocity $(1, 0)$ at $t = 0$, what is its position $\vec{r}(t)$ at time t?

3 Verify that the curve $\vec{r}(t) = \langle t\cos(t), 3t\sin(t), t^2 \rangle$ is located on the **elliptical paraboloid**

$$z = x^2 + \frac{y^2}{9} .$$

Use this fact to sketch the curve.

4 Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the curve obtained by intersecting the elliptical cylinder $x^2/9 + y^2/25 = 1$ with the surface $z = xy$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

5 Consider the curve

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle t^2, 1 + t, 1 + t^3 \rangle .$$

Check that it passes through the point $(1, 0, 0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

Lecture 8: Arc length and curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a parametrized curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then the number $L = \int_a^b |\vec{r}\,'(t)|\;dt$ is called the **arc length** of the curve.

We justify in class why this formula is reasonable. In space, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\;dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = \langle R\cos(t), R\sin(t)\rangle$ parameterized by $0 \le t \le 2\pi$ is 2π because the speed $|\vec{r}'(t)|$ is constant and equal to R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ has velocity $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1\rangle$ and constant speed $|\vec{r}\,'(t)| = |\langle -\sin(t), \cos(t), 1\rangle| = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2\rangle$$

for $1 \le t \le 2$? Answer: Because $\vec{r}\,'(t) = \langle 1, 1/t, t\rangle$, we have $\vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\;dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity".

4 Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 6t, t^3\rangle$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle$? Answer: We have $|\vec{r}\,'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi} (3/2)\sin(2t)\;dt = 6$.

6 Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ for $-1 \le t \le 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\;dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2}\;dx = 2\int_0^1 x\sqrt{1+x^2}\;dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = \langle t + \sin(t), \cos(t)\rangle$ parameterized by $0 \le t \le 2\pi$. We have $|\vec{r}'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\;dt$. A **substitution** $t = 2u$ gives $L = \int_0^{\pi} \sqrt{2 + 2\cos(2u)}\;2du = \int_0^{\pi} \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\;2du = \int_0^{\pi} \sqrt{4\cos^2(u)}\;2du = 4\int_0^{\pi} |\cos(u)|\;du = 8$.

8 Find the arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t)\rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1\;,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. Solution: We have $|\vec{r}\,'(t)| = \sqrt{1 + \sinh^2(t)} = \cosh(t)$ and $\int_{-1}^1 \cosh(t)\;dt = 2\sinh(1)$.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t\langle -\sin(t), \cos(t) \rangle$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\, dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ leads to the arc length integral $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\, dt$ which can only be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** of a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. Let $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = \langle t, f(t) \rangle$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

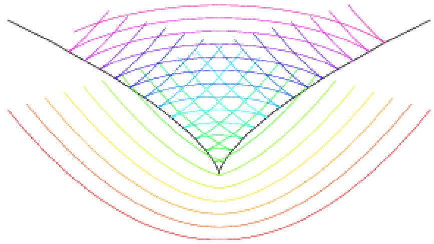
For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by N and $B$is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because $\vec{B}$ is automatically normal to $\vec{T}$ and $\vec{N}$, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

Here is an application of curvature: if a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call it the **evolute** of the original curve.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We prove this in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.

In this picture of John Harvard, software was looking for level curves for each color and started to draw at points where the curvature of the curves is large then follow the level curve.

Homework

1 a) Find the arc length of $\vec{r}(t) = \langle 4t^2, 4\sin(3t) - 4t\cos(t), 2\cos(3t) + 2t\sin(t)\rangle$, with $0 \le t \le \pi$.
b) Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ from $t = -1$ to $t = 1$.

2 Find the curvature of $\vec{r}(t) = \langle e^t\cos(t), e^t\sin(t), t\rangle$ at the point $(1, 0, 0)$.

3 Find the vectors $\vec{T}(t)$, $\vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = \langle t^2, t^3, 0\rangle$ for $t = 2$. Explore whether the vectors depend continuously on t for all t.

4 Let $\vec{r}(t) = \langle t, t^2\rangle$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$. It is known also as the **evolute** of the curve.

5 If $\vec{r}(t) = \langle -\sin(t), \cos(t)\rangle$ is the boundary of a coffee cup and light enters in the direction $\langle -1, 0\rangle$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.