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| Name: |
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- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-2, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
| 5      |  | 10  |
| 6      |  | 10  |
| 7      |  | 10  |
| 8      |  | 10  |
| 9      |  | 10  |
| Total: |  | 100 |

Problem 1) (20 points) No justifications are needed.

- 1) 

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 $\langle 0, 0, 1 \rangle \times (\langle 0, 1, 0 \rangle \times \langle 0, 1, 0 \rangle) = (\langle 0, 0, 1 \rangle \times \langle 0, 1, 0 \rangle) \times \langle 0, 1, 0 \rangle.$

**Solution:**

This is actually a counter example to the associativity property of the cross product. We have discussed that in problem session.

- 2) 

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 The unit tangent vector  $\vec{T}$  is perpendicular to the velocity vector  $\vec{r}'(t)$ .

**Solution:**

No, it is parallel to the velocity vector because it is just rescaled.

- 3) 

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 The point given in spherical coordinates as  $(\rho, \phi, \theta) = (2, \pi/2, \pi/2)$  is on the  $y$  axes.

**Solution:**

Yes,  $\phi = \pi/2$  means that we are on the equatorial plane.

- 4) 

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 On a slide shown in class, a T-shirt had the formula  $(x - a)^2 + (y - b)^2 = r^2$  with subtitle *I feel fabulously hyperbolic today!*

**Solution:**

No, the subtitle had been "going in circles".

- 5) 

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 The curvature of a circle of radius 3 is 3.

**Solution:**

No, it is  $1/3$ .

- 6) 

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 The triple scalar product of three vectors can not be negative because it computes volumes.

**Solution:**

No, it can come with a sign.

- 7) 

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| T | F |
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 If the dot product between two vectors is negative, then the two vectors form an obtuse angle.

**Solution:**

Yes, if  $\cos(\alpha) < 0$ , then this means that  $\alpha > \pi/2$ .

- 8) 

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 The equality  $|v|^2|w|^2 - (\vec{v} \cdot \vec{w})^2 = |\vec{v} \times \vec{w}|^2$  is called the Cauchy-Schwarz equality.

**Solution:**

No, it is called Lagrange identity or Cauchy-Binet formula. It actually implies Cauchy-Schwarz because it shows that the left hand side is bigger or equal to zero.

- 9) 

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| T | F |
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 The surface given in cylindrical coordinates as  $z^2 + r^2 = 1$  is a sphere.

**Solution:**

yes, it means  $x^2 + y^2 + z^2 = 1$ .

- 10) 

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 The arc length of the curve  $\langle \sin(t), \cos(t) \rangle$  from 0 to 1 is equal to 1.

**Solution:**

Yes, the speed is equal to 1 so that the integral is 1.

- 11) 

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 Three lines in space always intersect in at least one point.

**Solution:**

No, they can all be skew.

- 12) 

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 The curve  $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$  hits the plane  $z = 0$  at a right angle.

**Solution:**

The velocity vector is  $-1, 0, 1$ . This is not perpendicular to the plane.

- 13) 

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 The lines  $\vec{r}(t) = \langle t, -t, 2t \rangle$  and  $\langle 5 - t, 3 + t, -2t \rangle$  are parallel.

**Solution:**

Yes they have both the same velocity vector.

- 14) 

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 The parametrized curve  $\langle 2 \cos(t), 0, 3 \sin(t) \rangle$  is an ellipse.

**Solution:**

Indeed, it is part of the  $xz$ -plane.

- 15) 

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 If a cone is intersected with a plane, we always obtain an ellipse.

**Solution:**

No, other conic sections like parabola and hyperbola can occur

- 16) 

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 The vector  $\langle 3/5, 0, 4/5 \rangle$  is a direction.

**Solution:**

Yes, its length is equal to 1.

- 17) 

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 Two vectors  $\vec{v}$  and  $\vec{w}$  are parallel or anti-parallel if  $\vec{v} \times \vec{w} = \vec{0}$ .

**Solution:**

yes

- 18) ☐ T ☒ F  $\vec{u} \times (\vec{v} \times \vec{u}) = 0$  for all vectors  $\vec{u}, \vec{v}$ .

**Solution:**

Take  $u = i, v = j$ .

- 19) ☒ T ☐ F The plane parametrized by  $\vec{r}(t, s) = \langle t, s, 1 \rangle$  is the same than  $z = 1$ .

**Solution:**

Indeed

- 20) ☒ T ☐ F If  $f(x, y) = x^2y^2$ , then  $f_{xyxy} = 4$ .

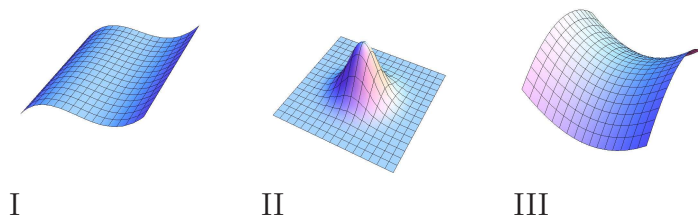
**Solution:**

Yes, use Clairaut

Total

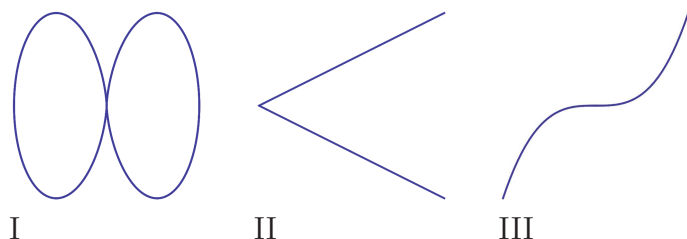
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the graphs of the functions  $f(x, y)$ . Enter O, if there is no match.



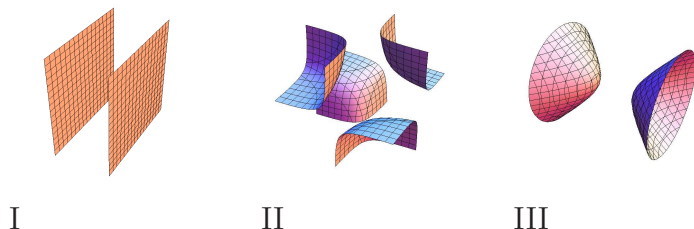
| Function $f(x, y) =$ | Enter O,I,II or III |
|----------------------|---------------------|
| $x^3 - y$            |                     |
| $x^2 - y^2$          |                     |
| $x^4$                |                     |
| $\exp(-x^2 - y^2)$   |                     |
| $\exp(-y^2)$         |                     |

b) (2 points) Match the plane curves with their parametrizations  $\vec{r}(t)$ . Enter O, if there is no match.



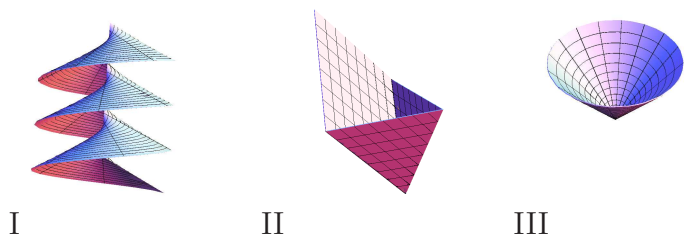
| Parametrization $\vec{r}(t) =$                            | O, I,II,III |
|-----------------------------------------------------------|-------------|
| $\vec{r}(t) = \cos^2(t) \langle \cos(t), \sin(t) \rangle$ |             |
| $\vec{r}(t) = \langle \cos^2(t), \sin^2(t) \rangle$       |             |
| $\vec{r}(t) = \langle t, t^3 \rangle$                     |             |
| $\vec{r}(t) = \langle  t , t \rangle$                     |             |

c) (2 points) Match functions  $g$  with level surface  $g(x, y, z) = 1$ . Enter O, if no match.



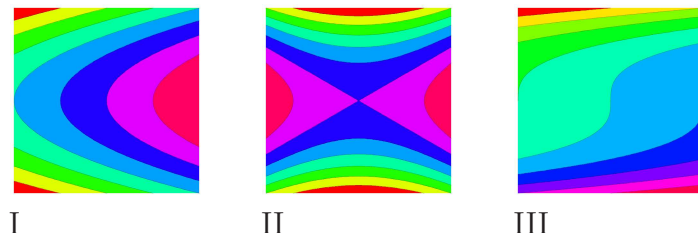
| Function $g(x, y, z) = 1$ | O, I,II,III |
|---------------------------|-------------|
| $x + y^2 + z^2 = 1$       |             |
| $xyz = 1$                 |             |
| $x^2 = 1$                 |             |
| $x^2 - y^2 - z^2 = 1$     |             |

d) (2 points) Match the parametrization. Enter O, where no match.



| $\vec{r}(s, t)$                                 | O,I,II,III |
|-------------------------------------------------|------------|
| $\langle t, t^3 - s^3, s \rangle$               |            |
| $\langle s^2 \cos(t), s^2 \sin(t), s^2 \rangle$ |            |
| $\langle s \cos(t), s \sin(t), t - s \rangle$   |            |
| $\langle  t ,  s ,  t + s  \rangle$             |            |

e) (2 points) Match the contour maps. Enter O, where no match.



| $f(x, y)$     | O,I,II,III |
|---------------|------------|
| $x - y^3$     |            |
| $x^2 + 3y^2$  |            |
| $-x^2 + 3y^2$ |            |
| $x - y^2$     |            |

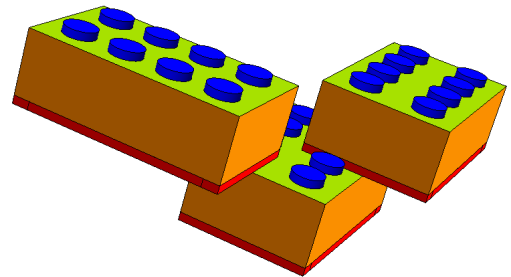
**Solution:**

- a) I, III, 0, II, 0
- b) I, 0, III, II
- c) 0, II, I, III
- d) 0, III, I, II
- e) III, 0, II, I

|                        |
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| Problem 3) (10 points) |
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Routine problems:

- a) (2 points)  $\langle 3, 1, 4 \rangle \cdot \langle 5, 9, 2 \rangle$ .
- b) (2 points)  $\langle 5, 5, 5 \rangle \times \langle 3, 2, 1 \rangle$ .
- c) (2 points)  $\langle 5, 5, 5 \rangle \cdot \langle 3, 2, 1 \rangle \times \langle 1, 1, 1 \rangle$ .
- d) (2 points)  $\vec{P}_{\vec{w}}(\vec{v})$  with  $\vec{v} = \langle 1, 1, 1 \rangle$  and  $\vec{w} = \langle 1, 2, 1 \rangle$ .
- e) (2 points)  $\cos(\alpha)$  of angle between  $\langle 1, 1, 1 \rangle, \langle 4, 3, 1 \rangle$ .

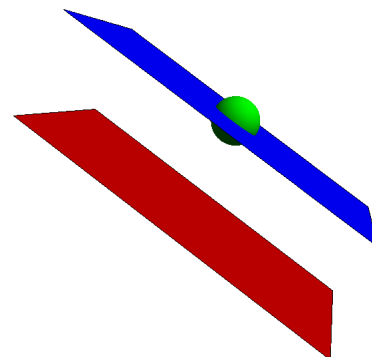


**Solution:**

- a) 32
- b)  $\langle -5, 10, -5 \rangle$ .
- c) 0 (two vectors are parallel)
- d)  $\langle 2, 4, 2 \rangle / 3$
- e)  $8/\sqrt{78}$ .

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| Problem 4) (10 points) |
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When two uncharged metallic parallel plates are put close together, there is an attractive force between them which can be explained by quantum field theory only. In May 14, 2013, an article suggested to use this Casimir effect for microchip designs. (Source Nature: <http://www.nature.com/ncomms/journal/v4/n5/full/nc>



a) (3 points) Locate a point  $P$  on the plane  $x + 2y + 2z = 4$ .

b) (7 points) Find the distance  $d$  between the plane  $x + 2y + 2z = 1$  and plane  $x + 2y + 2z = 4$ .

**Solution:**

a) There are many possibilities,  $P = (4, 0, 0), (0, 2, 0), (0, 0, 2), (0, 1, 1)$  were popular choices.

b) Compute the distance of  $P$  to the plane. The second plane has a point  $Q = (1, 0, 0)$ . Now compute  $d = |\vec{PQ} \cdot \vec{n}|/|\vec{n}| = 1$ , where  $\vec{n} = \langle 1, 2, 2 \rangle$ . It is also possible to compute the distance as  $|e - d|/|\vec{n}| = 3/3 = 1$  where  $e = 4$  is the constant in the first plane and  $d = 1$  is the constant in the second plane.

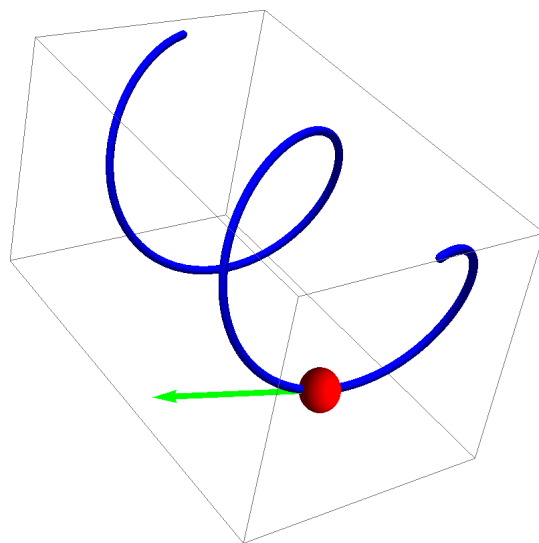
|                        |
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| Problem 5) (10 points) |
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Given  $\vec{r}(t) = \langle \sin(\pi t^2), t^2, \cos(\pi t^2) \rangle$ .

a) (4 points) Find the arc length from  $t = 0$  to  $t = 1$ .

b) (4 points) Find  $\vec{r}'(t)$  and  $\vec{r}''(t)$  at  $t = 1$ .

c) (2 point) What is the curvature at the point  $t = 1$ ?



**Solution:**

a)  $\vec{r}'(t) = \langle 2\pi t \cos(\pi t^2), 2t, -2\pi t \sin(\pi t^2) \rangle$ .

It has the length  $2t\sqrt{\pi^2 + 1}$ . When integrating from 0 to 1 we get  $\sqrt{\pi^2 + 1}$ .

b) The velocity evaluated at 1 is  $\langle -2\pi, 2, 0 \rangle$ , the acceleration computed at  $t = 1$  is  $\langle -2\pi, 2, 4\pi^2 \rangle$ .

c) Use  $|\vec{r}'(1) \times \vec{r}''(1)|/|\vec{r}'(1)|^3 = \pi^2/(\pi^2 + 1)$ .

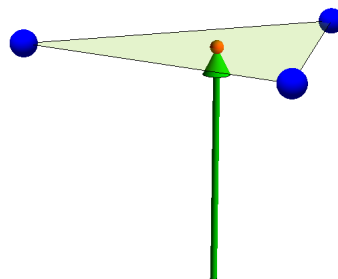
Problem 6) (10 points)

a) (2 points) Given three points  $A = (1, 1, 1)$ ,  $B = (1, 0, 1)$ ,  $C = (0, 1, 1)$ . Find the **center of mass**  $M = (A + B + C)/3$ .

b) (4 points) Find the equation  $ax + by + cz = d$  of the plane through  $A, B, C$ .

c) (4 points) Find a parametrization of the line through  $M$  which is perpendicular to the triangle.

Now you can put a needle on this line and place the triangle on top of it. It will float in equilibrium.



**Solution:**

a)  $(2, 2, 3)/3$

b)  $\vec{AB} \times \vec{AC} = \langle 0, 0, -1 \rangle$ . The equation is  $z = 1$ .

c)  $\vec{r}(t) = \langle 2, 2, 3 \rangle/3 + t\langle 0, 0, -1 \rangle = \langle 2/3, 2/3, 1 - t \rangle$ . There are of course other parametrizations which work too.

Problem 7) (10 points)

The **Pound-Rebka experiment** was one of the latest tests for Einstein's theory of relativity. The experiment, which measures the gravitational red shift had been installed in the 22.5-meter-high Jefferson Tower at the Harvard physics department just behind the Science Center. To celebrate this achievement, we climb to the top of the Jefferson tower and throw a stone from  $(0, 0, 25)$  towards the Science Center with initial velocity is  $\langle 10, 0, 20 \rangle$ . Where does the stone hit the ground, if the acceleration is  $\langle 0, 0, -10 \rangle$ ?



### Solution:

We integrate and use the initial conditions to fix the constants.

$$\vec{r}''(t) = \langle 0, 0, -10 \rangle.$$

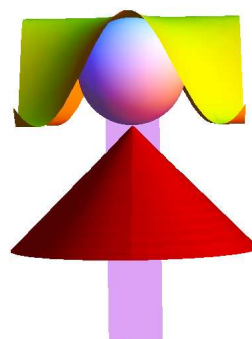
$$\vec{r}'(t) = \langle 20, 0, 5 - 10t \rangle$$

$\vec{r}(t) = \langle 10t, 0, 25 + 20t - 5t^2 \rangle$ . It hits at 5 seconds at the point  $(50, 0, 0)$ . If you look on a map, you see that the Science center is safe.

### Problem 8) (10 points)

We build an action figure for a future MMOG "World of Math21a". One of the figures is composed of four parametric surfaces. Parametrize these surfaces in the form  $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ . For example, to parametrize a cylinder  $x^2 + y^2 = 1$ , you would get  $\vec{r}(u, v) = \langle \cos(u), \sin(u), v \rangle$ . You do not have to give bounds for the parameters.

- (3 points) sphere  $(x - 1)^2 + (y - 2)^2 + z^2 = 9$ .
- (2 points) plane through  $(0, 0, 0), (1, 0, 0), (1, 1, 1)$ .
- (3 points) cone  $x^2 + z^2 = y^2$ .
- (2 points) graph of  $f(x, y) = \cos(xy)$ .



**Solution:**

We have to scale and translate the sphere, get two vectors in the plane, realize that  $c$  is a surface of revolution and just know how to parametrize a graph (which is difficult because it is tautological and so easy that one always missed it):

a)  $\vec{r}(u, v) = \langle 1 + 3 \cos(u) \sin(v), 2 + 3 \sin(u) \sin(v), 3 \cos(v) \rangle$ .

b)  $\vec{r}(u, v) = \langle u, v, v \rangle$ .

c)  $\vec{r}(u, v) = \langle v \cos(u), v, v \sin(u) \rangle$ .

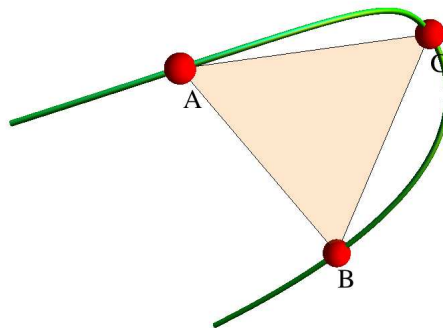
d)  $\vec{r}(u, v) = \langle u, v, \cos(uv) \rangle$ .

Problem 9) (10 points)

An asteroid moves on a curve  $\vec{r}(t) = \langle t, t^2, t^4 \rangle$ .

a) (5 points) Find the area of the triangle with vertices  $A = \vec{r}(-1)$ ,  $B = \vec{r}(1)$  and  $C = \vec{r}(0)$ .

b) (5 points) To find, whether the point  $D = \vec{r}(2) = \langle 2, 4, 16 \rangle$  is on the plane, compute the volume of the parallelepiped spanned by the four points.

**Solution:**

a)  $\vec{r}(0) = \langle 1, 0, 0 \rangle$   $\vec{r}(1) = \langle 2, 1, 1 \rangle$   $\vec{r}(-1) = \langle 0, 1, 1 \rangle$ . We have  $\vec{v} = \vec{r}(1) - \vec{r}(0)$  and  $\vec{w} = \vec{r}(-1) - \vec{r}(0)$  and  $\vec{n} = \vec{v} \times \vec{w} = \langle 1, 1, 1 \rangle \times \langle -1, 1, 1 \rangle = \langle 0, -2, 2 \rangle$ . The area of the triangle is one half of the area of the parallelogram which is  $\sqrt{4+4}/2 = 2\sqrt{2}/2 = \sqrt{2}$ .

b) We compute the volume of the parallel epiped spanned by  $\vec{AB}$ ,  $\vec{AC}$ ,  $\vec{AD}$ . The answer is 24. This actually shows that the four points are not part of a plane.