7/7/2016 FIRST HOURLY Practice V **Maths 21a, O.Knill, Summer 2016**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-3 give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F — The length of the vector $\vec{v} \times \vec{v}$ is 3, if $\vec{v} = \langle 1, 1, 1 \rangle$.

Solution:
The length is zero. The dot product is 3 not the cross product.

2) T F — For any 4 points A, B, C, D in space, there is a plane which goes through these points.

Solution:
We can not manage 4 points in general. The distance of a forth point to the plane defined by the other three is in general positive.

3) T F — For any two points P, Q in space which have positive distance there is exactly one plane which has equal distance to both points.

Solution:
There is unique one with minimal distance: it is the plane perpendicular to PQ which passes through the midpoint $(P+Q)/2$. There are however also other planes parallel to the two points or even planes through the two points for which the distance is zero.

4) T F — The graph of a function $f(x, y)$ can be parametrized as $\vec{r}(x, y, z) = z - f(x, y)$.

Solution:
This is not a parametrization

5) T F — The graph of the function $f(x, y) = x^2 - y^2$ is called a hyperboloid.

Solution:
Nope, it is a hyperbolic paraboloid.

6) T F The equation $\rho\cos(\theta)\sin(\phi) = 1$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, we have $x = \rho\cos(\theta)\sin(\phi)$.

7) T F The vector $\langle 1, 2\rangle$ is perpendicular to the line $x + 2y = 2$.

Solution:
It is the gradient $\langle 1, 2\rangle$.

8) T F The triple scalar product between the vectors $\vec{i}$, $\langle 0, 2, 0\rangle$ and $\langle 1, 1, 1\rangle$ is 2.

Solution:
Compute it

9) T F The two parametrized curves $\vec{r}(t) = \langle t, t^2, t^6\rangle, 0 \le t \le 2$ and $\vec{R}(t) = \langle t^2, t^4, t^{12}\rangle, 0 \le t \le 2$ have the same curvature at $t = 1$.

Solution:
This is a change of parametrization.

10) T F The point $(x, y, z) = (1, 1, 1)$ has the spherical coordinates $(\rho, \theta, \phi) = (\sqrt{3}, \pi/2, \pi/4)$.

Solution:
Apply the transformation formulas.

11) T F The distance between the cylinders $x^2 + z^2 = 1$ and $x^2 + (y - 3)^2 = 1$ is 1.

Solution:
The two cylinders intersect.

12) T F If P is the projection onto the line spanned by a vector $\vec{w}$ then $\mathrm{P}(\vec{w}) = \vec{w}$.

Solution:
Projecting a vector in the line onto the line does not change it.

13) T F The vector projection of $\langle 2, 3, 1\rangle$ onto $\langle 1, 1, 1\rangle$ is parallel to $\langle 1, 1, 1\rangle$.

Solution:
Yes.

14) T F The triple scalar product has the property $\vec{u} \cdot (\vec{v} \times \vec{u}) = 0$ for any two vectors $\vec{u}, \vec{v}$.

Solution:
The volume of the parallelepiped spanned by two vectors only is 0.

15) T F For any two vectors $\vec{v}$ and $\vec{w}$, the length of the cross product $\vec{v} \times \vec{w}$ is equal to the length of $\vec{v} + \vec{w}$.

Solution:
This is utter nonsense.

16) T F If the domain of a function is the entire plane, then the range of the function is the entire line or a half line.

Solution:
Already $f(x, y) = x^2 + y^2 + 1$ is a simple counter example.

17) T F There are two unit vectors $\vec{v}$ and $\vec{w}$ which have the property that $\vec{v} \cdot \vec{w} = \pi$.

Solution:
We just need the angle to be right. But the dot product is always smaller or equal than 1 by Cauchy

18) | T | F | If we travel with unit speed on a street, then the curvature $\kappa(\vec{r}(t))$ of the street is bounded by the magnitude of the acceleration $|\vec{r}\,''(t)|$.

Solution:
Use the formula $|\vec{r}\,''(t) \times \vec{r}|$ if $\vec{r}\,'$ is a unit vector.

19) | T | F | The curve $\vec{r}(t) = \langle \cos(2t)\sin(3t), \sin(2t)\sin(3t), \cos(3t) \rangle$ is located on a sphere.

Solution:
Check $x^2 + y^2 + z^2 = 1$.

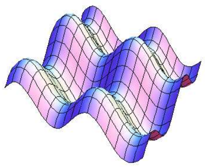
20) | T | F | The surface $x^2 - y^2 + z^2 = 4z - 3$ is a one sheeted hyperboloid.

Solution:
Complete the square.

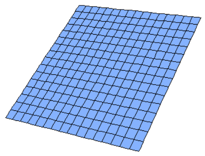
[ ] Total

Problem 2) (10 points) No justifications are needed in this problem.

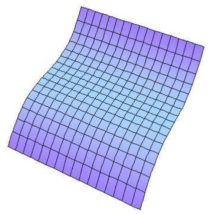
a) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.



I



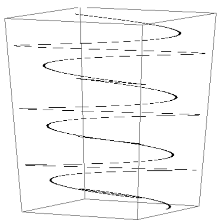
II



III

Function $f(x,y) =$	Enter O,I,II or III
y^3	
$\sin(x) + \sin(y)$	
$x^2 - y^2$	
$x^4 - y^4$	
$x + y$	

b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



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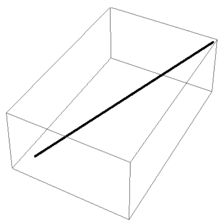

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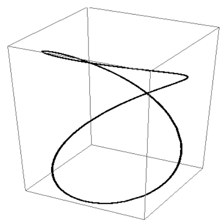
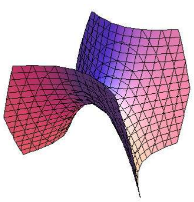


III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\vec{r}(t) = \langle \cos(2t), \sin(3t), t \rangle$	
$\vec{r}(t) = \langle \lvert t \rvert, \lvert t \rvert, t^2 \rangle$	
$\vec{r}(t) = \langle \sin(2t)/2, \sin^2(t), \cos(t) \rangle$	
$\vec{r}(t) = \langle 1 + 2t, 2 + 3t, t \rangle$	

c) (2 points) Match the functions g with the level surface $g(x,y,z) = 1$. Enter O, where no match.



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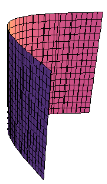

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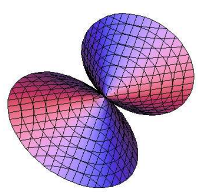
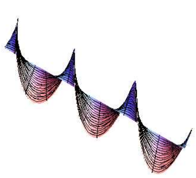


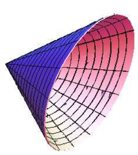
III

Function $g(x,y,z) = 1$	O, I,II,III
$g(x,y,z) = x^2 + z = y^2$	
$g(x,y,z) = x^2 - y^2 = 1$	
$g(x,y,z) = x^2 - y^2 + z^2 = 0$	
$g(x,y,z) = \frac{x^2}{2} + 2y^2 + z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.



I



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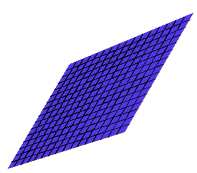


III

Function $g(x,y,z) =$	O,I,II,III
$\vec{r}(s,t) = \langle t, s, ts \rangle$	
$\vec{r}(s,t) = \langle t - 1, s, t + s \rangle$	
$\vec{r}(s,t) = \langle t, s\cos(t), s\sin(t) \rangle$	
$\vec{r}(s,t) = \langle s, s\cos(t), s\sin(t) \rangle$	

Solution:
a) III,I,0,0,II
b) I,O,III,II
c) I,II,III,0
d) 0,III,I,II

Problem 3) (10 points) No detailed justifications are needed.

a) (6 points) We are given a point P, a line L determined by A and $\vec{v}$ a line M determined by B and $\vec{w}$ and a plane Σ. Match the distances with distance formulas and possible explanations:

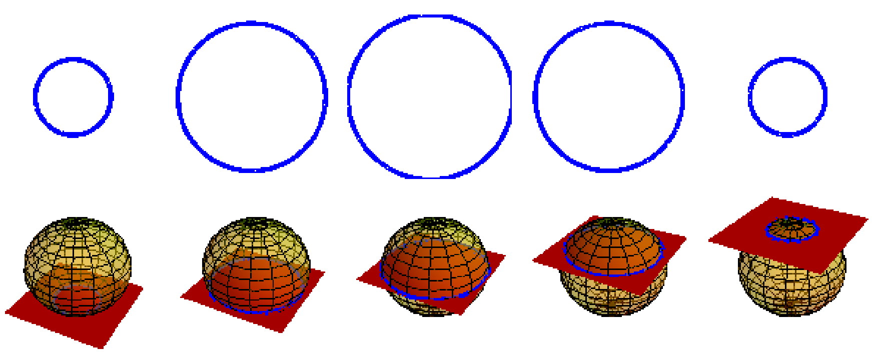
distance	the formula involves (A-C)	and can be explained by (U-W)
$d(P,\Sigma)$		
$d(L,M)$		
$d(P,L)$		

A) triple scalar product
B) dot product and no cross product
C) cross product only

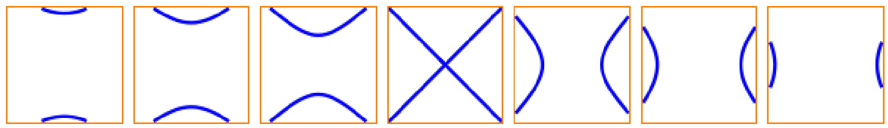
U projection of a connection vector
V volume of parallelepiped divided by base area
W area of parallelogram divided by base length

b) (4 points total) Edwin Abbott's novella "Flatland, A Romance of Many dimensions" deals with "flatlanders" who can not perceive three dimensional objects. By looking at various traces $f = c$ a flatlander still can see a three dimensional object. For example, a three dimensional sphere is visualized as follows.

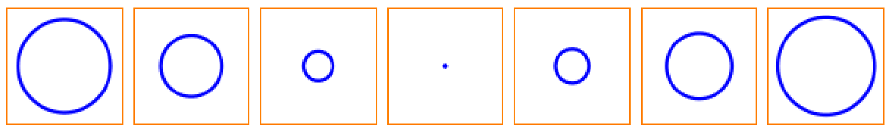
Edwin Abbott: 1838-1926



b1) (2 points) We are in flatland, where multivariable calculus students only work with 2 dimensions. To get a feel for the third dimension the following sequence of generalized traces are shown with a frame around each picture. They illustrate the graph of a function of two variables. Which surface is under consideration?



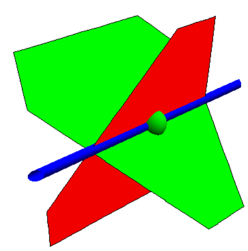
b2) (2 points) In the first midterm exam of that multivariable calculus course, the class has to find the name of the surface of the following sequence of traces. Can you do it?



Solution:
a) B, A, C and U, V, W.
b) It is a hyperbolic paraboloid.
c) it is a double cone.

Problem 4) (10 points)

Find the equation $ax + by + cz = d$ of a plane Σ perpendicular to the planes $x + y - z = 1$ and $x + 2y - 3z = 0$ so that the surface Σ contains the point $P = (2, 1, 2)$.



Solution:
Take the cross product of the normal vectors $\langle 1, 1, -1 \rangle$ and $\langle 1, 2, -3 \rangle$ which is $\langle -1, 2, 1 \rangle$. This vector is in the intersection line and therefore perpendicular to the two planes. The plane normal is $-x + 2y + z = 2$.

Problem 5) (10 points)

In this problem we find some parametrizations of surfaces which is of the form

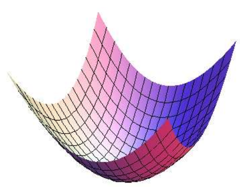
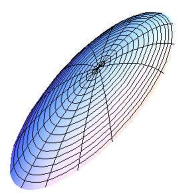
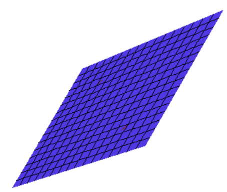
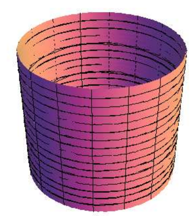
$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$

a) (2 points) Parametrize the **paraboloid** $z = x^2 - y^2$.

b) (3 points) Parametrize (the entire!) **ellipsoid** $(x-1)^2 + \frac{(y-2)^2}{4} + z^2 = 1$.

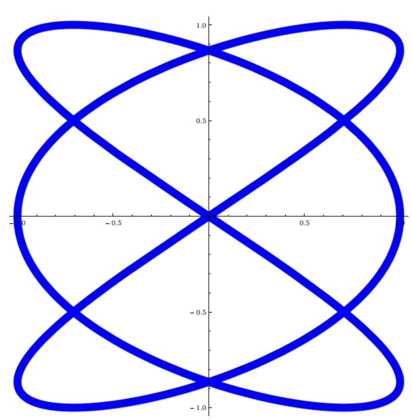
c) (2 points) Parametrize the **plane** $x + y + z = 3$.

d) (3 points) Parametrize the **cylinder** $x^2 + z^2 = 1$.

Solution:
a) $\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$. This is a graph.
b) $\vec{r}(u,v) = \langle 1 + \cos(u)\sin(v), 2 + 2\sin(u)\sin(v), \cos(v) \rangle$. In this problem, a graph parametrization like $\langle u, v, f(u,v) \rangle$ would not give the entire ellipsoid.
c) $\vec{r}(u,v) = \langle 3,0,0 \rangle + u\langle 3,-3,0 \rangle + v\langle 3,0,-3 \rangle$. There are of course many possibilities here. An other simple solution is $\langle u, v, 3-u-v \rangle$.
d) $\vec{r}(u,v) = \langle \cos(u), v, \sin(u) \rangle$. In this problem, the order was often incorrect and the standard cylinder $\langle \cos(u), \sin(u), v \rangle$ along the z-axes taken.

Problem 6) (10 points)



a) (6 points) Find the velocity vectors $\vec{r}\,'(\pi/2)$ and $\vec{r}\,'(3\pi/2)$. Now find the angle between these vectors at the point, where the curve

$$\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$$

self-intersects at the origin.
b) (4 points) Find $\vec{r}\,'(\pi/2), \vec{r}\,''(\pi/2)$ and $\vec{r}\,'(\pi/2) \times \vec{r}\,''(\pi/2)$. What is the curvature $|\vec{r}\,'(\pi/2) \times \vec{r}\,''(\pi/2)|/|\vec{r}\,'(\pi/2)|^3$ at $(0,0)$?

Solution:
a) $\vec{r}'(t) = \langle -3\sin(3t), 2\cos(2t)\rangle$ so that $\vec{r}'(\pi/2) = \langle 3, -2\rangle$ and $r'(3\pi/2) = \langle -3, -2\rangle$. We have $\cos(\alpha) = -5/13$.
b) $\vec{r}''(t) = \langle -9\cos(3t), -4\sin(2t)\rangle$. We have $\vec{r}''(\pi/2) = \langle 0, 0\rangle$. The curvature is therefore zero.

Problem 7) (10 points)

Compute the following expressions:

a) (2 points) the length of the vector $\langle 12, 5\rangle$,

b) (2 points) the cross product $\langle 1, 2\rangle \times \langle 5, 1\rangle$,

c) (2 points) the dot product $\langle 1, 4\rangle \cdot \langle 2, 2\rangle$,

d) (2 points) the projection $P_{\vec{w}}(\langle 1, 2\rangle)$ onto the x axes spanned by $\vec{w} = \langle 1, 0\rangle$.

e) (2 points) the angle between $\langle 1, 1\rangle$ and $\langle 0, 1\rangle$.

Solution:
a) 13
b) -9. Notice that the cross product between two vectors in the plane was defined as a scalar!
c) 10
d) $\langle 1, 0\rangle$.
e) $\alpha = \pi/4$.

Problem 8) (10 points)

Oliver attempted to build some of Chihuly's sculptures in Mathematica. To get the organic forms, spiral curves of the form

$$\vec{r}(t) = \langle \sin(t)\cos(t), \sin(t)\sin(t), t \rangle$$

were used. Find the arc length of this curve from $0 \leq t \leq \pi$.

Solution:
Using double angle formulas we have $\vec{r}'(t) = \langle \cos(2t), \sin(2t), 1 \rangle$ which has length $|\vec{r}'(t)| = \sqrt{2}$. The arc length is

$$L = \int_0^\pi \sqrt{2}\, dt = \pi\sqrt{2} \; .$$

Remark: When not using double angle formulas, the computation would be a bit more complicated $\vec{r}'(t) = \langle \cos^2(t) - \sin^2(t), 2\cos(t)\sin(t), 1 \rangle$. Summing the squares of the components would give $\cos^4(t) + 2\cos^2(t)\sin^2(t) + \sin^4(t) + 1$ which can be written as $[(\cos^2(t) + \sin^2(t)]^2 + 1 = 2$.

Problem 9) (10 points)

To celebrate **Bastille day** today on the 14th of July, we fire some fireworks. The rocket experiences during the firing phase a time dependent acceleration

$$\vec{r}\,''(t) = \langle 0, 0, 100\cos(t) - 10 \rangle$$

during the rocket firing time $[0, \pi/2]$. It lifts off at $\vec{r}(0) = \langle 1, 0, 0 \rangle$ with zero velocity $\vec{r}\,'(0) = \langle 0, 0, 0 \rangle$. Where will the rocket be at time $t = \pi/2$?

Solution:
Integrate the expression $\vec{r}(t)$ with respect to t to get

$$\vec{r}'(t) = \langle 0, 0, 100 \sin(t) - 10t \rangle + \vec{c} \,.$$

Comparing this with $t = 0$ shows that $\vec{c} = \vec{0}$. Now integrate again:

$$\vec{r}(t) = \langle 0, 0, -100 \cos(t) - 5t^2 \rangle + \vec{c}$$

Comparing with $t = 0$ shows that $\vec{c} = \langle 1, 0, 100 \rangle$. We get

$$\vec{r}(t) = \langle 1, 0, 100 - 100 \cos(t) - 5t^2 \rangle \,.$$

At time $t = \pi/2$, this is $\langle 1, 0, 100 - 5\pi^2/4 \rangle$.
Remark: most mistakes here were done by matching the initial condition and a result without 100 achieved.