Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The function $f(x,y) = x^6 + y^6$ has exactly one minimum.

Solution:
There is only one critical point and it is a minimum.

2) | T | F | The gradient $\nabla f(1,1,1)$ is tangent to the surface $x^2 + y^2 - z^2 = 1$.

Solution:
It is perpendicular

3) | T | F | The directional derivative $D_{\vec{v}} f(0,0)$ is positive if the unit vector $\vec{v}$ and the vector $\nabla f(0,0)$ form an acute angle.

Solution:
Yes, in that case the dot product of $\vec{v}$ and $\nabla f(0,0)$ which is the directional derivative is positive.

4) | T | F | If $f_x(0,0) = f_y(0,0) = 0$ and $f_{xx}(0,0) > 0, f_{yy}(0,0) > 0, f_{xy}(0,0) < 0$, then $(0,0)$ is a local minimum of f.

Solution:
It does not imply $D > 0$.

5) | T | F | If $f_x(0,0) = f_y(0,0) = 0$ and $f_{xx} > 0, f_{yy}(0,0) = 0, f_{xy}(0,0) < 0$, then $(0,0)$ is a saddle point.

Solution:
The information $f_{xx} > 0, f_{yy} > 0$ is not necessary. $f_{xy} < 0$ and the critical point property suffice.

6) T **F** The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \nabla f(\vec{r}\,'(t))$ for any function $f(x, y, z)$ and any curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Solution:
The second part should be the velocity

7) T **F** The point $(0, 1)$ is a maximum of $f(x, y) = x + y$ under the constraint $g(x, y) = x = 1$.

Solution:
The gradients are not parallel.

8) **T** F The function $u(x, y) = \sqrt{x^2 + y^2}$ solves the partial differential equation $u_x^2 + u_y^2 = 1$.

Solution:
Just differentiate.

9) **T** F Let $f(x, y) = x^3 y^2$. At $(0, 0)$ and every direction $\vec{v}$ we have $D_{\vec{v}} f(0, 0) = 0$.

Solution:
The point is a critical point.

10) T **F** The identity $f_{xyx} = f_{yxy}$ holds for all smooth functions $f(x, y)$.

Solution:
Clairaut's theorem does not imply that as there are 2 derivatives with respect to x and 2 derivatives with respect to y.

11) **T** F The integral $\int_0^x \int_0^y s^2 t^2 \, ds dt$ is positive if $x > 0$ and $y > 0$.

Solution:
It is a volume and not a signed volume because the function $s^2 t^2$ is positive.

12) T **F**

If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \times \vec{r}(u,v) = 0$.

Solution:
There is no relation between $\vec{r}(u,v)$ and the gradient ∇f. There would be relations between the partial derivatives $\vec{r}_u(u,v), \vec{r}_v(u,v)$ although.

13) **T** F

We have $|D_{\langle 1,0\rangle} f(0,0)| \leq |\nabla f(0,0)|$.

Solution:
By the cos-formula.

14) T **F**

Any smooth function $f(x,y)$ has a critical point inside the region $0 \leq x^2 + y^2 < 1$.

Solution:
Take $f(x,y) = x$ for example. It does not have any critical point in the interior. There are extrema on the closed disc, but these are not critical points of f as the gradient of f is not zero there.

15) **T** F

The surface area of a surface $\vec{r}(u,v) = \langle 2u, 3v, 0\rangle$ with (u,v) in a disk $R = \{u^2 + v^2 \leq 1\}$ is given by the integral $\int\int_R 6\, dudv$.

Solution:
Yes, $|\vec{r}_u \times \vec{r}_v| = 6$.

16) T **F**

The Lagrange multiplier λ of a critical point of $f(x,y)$ is positive if the function $f(x,y)$ is positive there.

Solution:
The sign of λ has nothing to do with f. It also depends on g. For example, if g changes sign, then the sign of λ changes too.

17) **T** F

The equation $f_{xy} = f^2 f_x + f_y$ is an example of a partial differential equation.

Solution:
Yes, as it is an equation for a function f involving partial derivatives.

18) T F If $(0,0)$ is a critical point of $f(x,y)$ and $f_{xy}(0,0) = 0$ and $f_{xx}(0,0)$, $f_{yy}(0,0)$ have different signs, then $(0,0)$ is a saddle point.

Solution:
It does not apply that $(0,0)$ is a critical point.

19) T F If $f(x,y)$ has the critical point $(0,0)$, then $(f(x,y))^4$ has the critical point $(0,0)$ too.

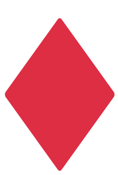
Solution:
The gradient is $\langle 3f^3 f_x, 3f^3 f_y \rangle$

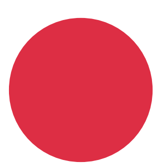
20) T F If $f(x,y)$ is the linear function $f(x,y) = 3x+y-3$, then $g(x,y) = |\nabla f(x,y)|$ is constant.

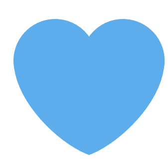
Solution:
The gradient is constant.

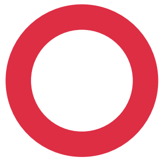
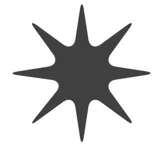
Problem 2) (10 points) No justifications are needed

a) (6 points) Some hate them, some love them. Any way, last Monday was **Emoji day**. Match the following Emoji with the integrals. There is an exact match.

A 

B 

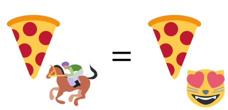
C 

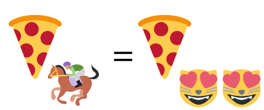
D    E    F 

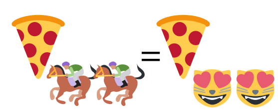
Enter A-F	Area integral
	$\int_0^{2\pi} \int_0^{1+\cos^6(4\theta)} \; r dr d\theta$
	$\int_0^{2\pi} \int_0^{(\theta-\pi/2))^2} \; r dr d\theta$
	$\int_{-1}^{1} \int_{\lvert x\rvert-1}^{1-\lvert x\rvert} 1 \; dxdy$
	$\int_0^{2\pi} \int_0^{1} r \; dr d\theta$
	$\int_0^{2\pi} \int_0^{1+\cos^4(5\theta/2-1)} \; r dr d\theta$
	$\int_0^{2\pi} \int_3^{4} r \; dr d\theta$

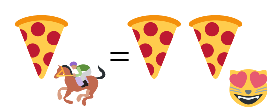
b) (4 points) Match the following **partial differential equations**. The pizza Emoji is a function and the rest of the Emoji's are variables. We transport hamburgers in the heat wave:

Fill in a)-d)	Name
	Transport
	Burgers
	Heat
	Wave


a)


b)


c)

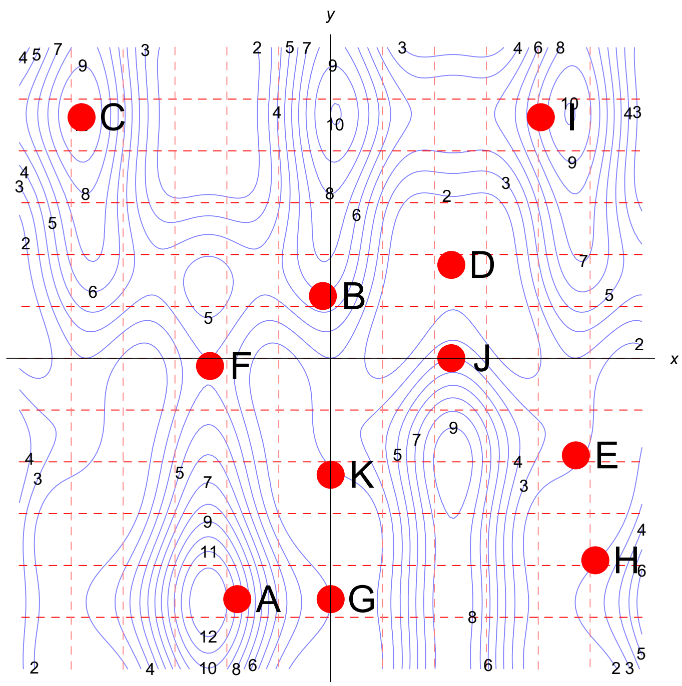

d)

Solution:
a) FCABED
b) ADBC

Problem 3) (10 points) (No justifications are needed.)

a) (5 points) In each part, pick the correct point in $A - K$. There is an answer to each case. We don't tell you whether a point can appear twice.

	Choose one A-K
A point where $f_x = 0$, $f_{xx} < 0$ and $f_y > 0$	
A point where $f_y = 0$, $f_{yy} < 0$ and $f_x > 0$	
A local maximum	
A saddle point	
A local maximum under the constraint $x = 0$	



b) (5 points) Fill in the missing parts of the theorems or results:

i) Assume $f_x(0,0) = 0, f_y(0,0) = 0$: If $D(0,0) > 0$ and $f_{xx}(0,0) < 0$ the $(0,0)$ is a local [ ] .

ii) The gradient $\nabla f(0,0)$ is [ ] to the level curve $f(x,y) = d$ where d is the constant $d = f(0,0)$.

iii) For all continuous functions $f(x,y)$ the identity $\int_2^3 \int_5^8 f(x,y)\ dydx\ \int_5^8 \int_2^3 f(x,y)\ dxdy$ is assured by the [ ] theorem.

iv) The surface area of a parametrized surface $\vec{r}(u,v)$ parametrized over a region R is $\int\int_R$ [ ] $dudv$.

v) If $(0,0)$ is not a critical point of f, then the directional derivative of f in the direction $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$ is known to be [ ].

Solution:
a) BICFG
b) Maximum, perpendicular, Fubini, $|\vec{r}_u \times \vec{r}_v|$, positive.

Problem 4) (10 points)

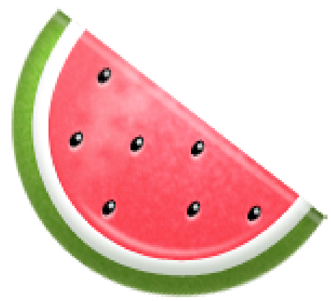
The **Emoji** with hex number 1F4C6 is a melon shaped candy. The outer radius is x, the inner is y. Assume we want to maximize the **sweetness function**

$$f(x,y) = x^2 - 2y^2$$

under the constraint that

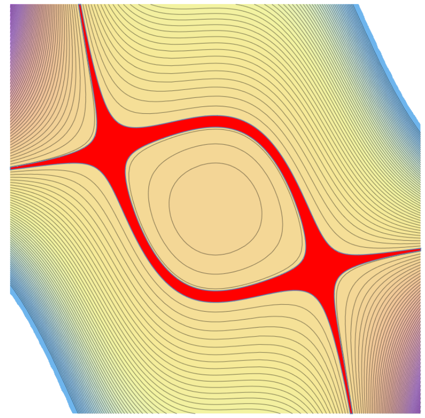
$$g(x,y) = x - y = 2 \ .$$

Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method!



Solution:
The gradients of f and g are $\nabla f = \langle 2x, -2y \rangle, \nabla g = \langle 1, -1 \rangle$. The Lagrange equations are $2x = \lambda, -2y = -\lambda$. Elimination λ gives $x = 2y$. Plug this into the constraint. We get $\boxed{x = 4, y = 2}$ as the maximum.

Problem 5) (10 points)



We go into the Emoji **design business**. But instead of using the Google Emoji online generator written for children ("code your own Emoji character"), we use Math and level curves. Lets take the function

$$f(x,y) = x^2 + 2x^3y + y^2 .$$

Find and classify all the critical points.

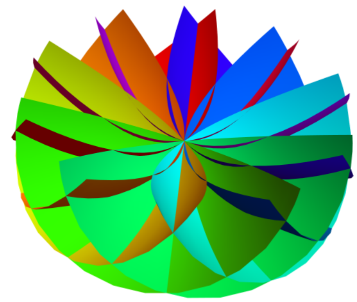
The picture shows the regions $0.23 < f(x,y) < 0.3$. We don't know what **emotion** this Emoji should represent, but it just somehow **looks cool**.

Solution:
The critical points are obtained by solving $\nabla f = \langle 2x + 6x^2y, 2x^3 + 2y \rangle = \langle 0, 0 \rangle$. Factoring out the first equation gives $x(2 + 6xy) = 0, 2x^3 + 2y = 0$. There are two possibilities for the first equation. Either $x = 0$ or $y = -x/3$. In the first case we get from the second equation $y = 0$. In the second case we get from the second equation $y = -x^3$ which gives from the first equation $2x - 6x^5 = 0$ showing $x = \pm 3^{-1/4}$. There are three solutions therefore:

(x,y)	D	f_{xx}	nature	value
$(0,0)$	4	2	minimum	0
$-3^{-1/4}$	$3^{-1/4}$	-16	-2 saddle	$3\sqrt{3}$
$3^{-1/4}$	$-3^{-1/4}$	-16	-2 saddle	$3\sqrt{3}$

Problem 6) (10 points)

We build a **paper origami** using 10 surface leafs. The picture is seen to the right. We look at one leaf only.



Find the surface area of this surface parametrized by

$$\vec{r}(u,v) = \langle u+v, u-v, u^2+v^2 \rangle ,$$

with $u^2 + v^2 \leq 4, v > 0$.

Solution:
We compute $\vec{r}_u \times \vec{r}_v = \lambda 2u + 2v, 2u - 2v, -2\rangle$. It length is $|\vec{r}_u \times \vec{r}_v| = \sqrt{4 + 8u^2 + 8v^2}$. We have now to compute the double integral of this function over the half disc R of radius 2. We use polar coordinates and get

$$\int_0^{\pi} \int_0^2 \sqrt{4+8r^2} r \; dr d\theta .$$

A substitution like $u = 4 + 8r^2, du = 16rdr$ gives us the answer $\boxed{26\pi/3}$. A common mistake was to overlook that $v > 0$ was assumed which gave an integration from 0 to π rather than from 0 to 2π.

Problem 7) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}}$ = 262537412640768743.99999999999925... is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

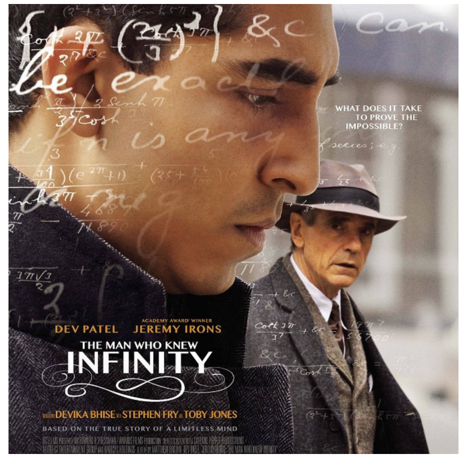
Let

$$f(x,y) = x\sqrt{y} \ .$$

Estimate

$$f(3.141,163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



Ramanujan is featured in the movie: "The Man who knew infinity", 2015

Solution:
$\nabla f(x,y) = \langle \sqrt{y}, x/(2\sqrt{y}\rangle$ which is $\langle 13, 3/(2*13)\rangle = \langle 13, 3/26\rangle$. Now $L(3.141,163) = 39 + 0.141*13 - 18/(2*13) = \boxed{40.1407}$. The actual result is 40.1016.

Problem 8) (10 points)

a) (5 points) Find the tangent plane to the surface

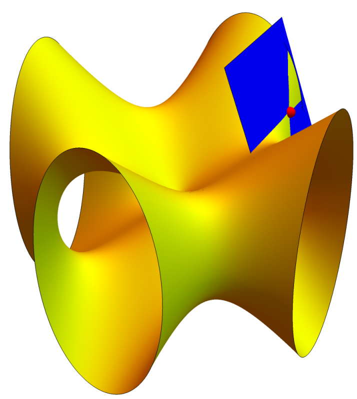
$$x^2 + y^2 - x^2y^2 - z^2 = 0$$

at the point $(x,y,z) = (1,2,1)$.

b) (5 points) Find the tangent line to the curve

$$x^2 + y^2 - x^2y^2 = -23$$

at the point $(x,y) = (3,2)$.

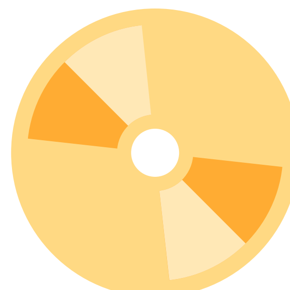


Solution:
a) $\nabla f = \langle 2x - 2xy^2, 2y - 2x^2y, -2z \rangle$. At the point $1, 2, 1$ it is $\langle -6, 0, -2 \rangle$. The equation of the plane is $\boxed{3x + z = 4}$.
b) $\nabla f = \langle 2x - 2xy^2, 2y - 2x^2y \rangle = \langle -18, -32 \rangle$. The equation of the line is $\boxed{9x + 16y == 59}$, where the constant 59 was obtained by plugging in the point $(x, y) = (3, 2)$.

Problem 9) (10 points)

a) (5 points) Integrate

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\ dxdy\ .$$



b) (5 points) Integrate the double integral

$$\int \int_R (x^2 + y^2)^{5/2}\ dxdy\ ,$$

where R is the region $1 \leq x^2 + y^2 \leq 4$ and $x \leq 0, y \geq 0$.

Solution:
a) This is a typical problem where we have to change the order of integration. Switch to a type 1 integral (make a picture) to get the integral $\int_0^{\sqrt{pi}} \int_0^x \sin(x^2)\ dydx$ which can now be solved by a simple substitution to be $\boxed{1}$.
b) This is a problem, where we have to change to polar coordinates. The integral is

$$\int_0^{\pi/2} \int_1^2 r^5 r\ drd\theta$$

which gives $\boxed{127\pi/14}$.