Archimedes' Revenge

Mark B. Villarino Escuela de Matemática, Universidad de Costa Rica,
10101 San José, Costa Rica

1 Introduction

Some 2300 years ago the great Archimedes proved that the volume of the orthogonal intersection of *two* circular cylinders of equal radius, today called a *bicylinder*, is *two thirds of the volume of its circumscribed cube.* ([1])(It seems have remained unnoticed that if the two cylinders intersect at an *angle,* then the volume of the solid is *two thirds that of the circumscribed box.*) Later, modern mathematicians found the volume of the intersection of *three* such cylinders, a so-called *tricylinder* or "Steinmetz" solid, and its computation has been the cause of countless desperation headaches in calculus students.

Then, in 2017, Oliver Knill proposed, under the title "Archimedes' Revenge," that one prove that *the volume of the intersection,* $\mathbf{R}$, *of the three (solid) hyperboloids*

$$x^2 + y^2 - z^2 \leqslant 1; y^2 + z^2 - x^2 \leqslant 1; z^2 + x^2 - y^2 \leqslant 1$$

is equal to $\mathbf{ln(256)}$.

We offer such a proof. We point out that it is motivated by the one due to Knill's student, Runze Li, published by the Harvard Maths 21A course in 2017.

2 The complete volume in the first octant.

By symmetry, the intersection, $\mathbf{R}$, of the three hyperboloids is the union of eight congruent solids, one in each octant. *We will compute the volume of the component solid,* $\mathbf{R_1}$, *in the first octant.* The complexity of the solid $\mathbf{R}_1$

is shown by the brute-force triple integral for its volume, namely:

$$
\begin{aligned}
R_1 \;=\;& \int_0^{\frac{1}{\sqrt{2}}} \left\{ \int_0^y \int_0^{\sqrt{x^2+y^2}} + \int_y^{\frac{1}{\sqrt{2}}} \int_{\sqrt{y^2-x^2}}^{\sqrt{x^2+y^2}} + \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{2y^2+1}}{\sqrt{2}}} \int_{\sqrt{x^2-y^2}}^{\sqrt{1-x^2+y^2}} \right\} 1 \; dy \; dx \; dz \\
&+ \int_{\frac{1}{\sqrt{2}}}^{\sqrt{\frac{3}{2}}} \left\{ \int_{\frac{\sqrt{2y^2-1}}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{\sqrt{y^2-x^2}}^{\sqrt{1+x^2-y^2}} + \int_{\frac{1}{\sqrt{2}}}^{y} \int_{\sqrt{x^2+y^2-1}}^{\sqrt{1+x^2-y^2}} + \int_y^1 \int_{\sqrt{x^2+y^2-1}}^{\sqrt{1-x^2+y^2}} \right\} 1 \; dy \; dx \; dz \\
&+ \int_{\sqrt{\frac{3}{2}}}^{1} \left\{ \int_{\frac{\sqrt{2y^2-1}}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{\sqrt{y^2-x^2}}^{\sqrt{1+x^2-y^2}} + \int_{\frac{1}{\sqrt{2}}}^{y} \int_{\sqrt{x^2+y^2-1}}^{\sqrt{1+x^2-y^2}} + \int_y^1 \int_{\sqrt{x^2+y^2-1}}^{\sqrt{1-x^2+y^2}} \right\} 1 \; dy \; dx \; dz.
\end{aligned}
$$

This is by no means the end of the story. The first two integrals in the second and third lines cannot be computed directly, but one must change the order of integration to calculate them. Thus a complete computation of the full volume, R_1, is extraordinarily daunting and tedious.

3 Symmetry

We will show how symmetry considerations reduce the computation of R_1 to *one single integral!*

The solid $\mathbf{R_1}$ is composed of *five* pieces:

- *two back-to-back tetrahedra* Π_1 and Π_2 with a common base, namely the equilateral triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ and opposite vertices $(0,0,0)$ and $(1,1,1)$, respectively.
- *three congruent curved pieces* each of which is bounded by a face of the larger tetrahedron, Π_2, and an hyperboloid.

4 Volume of the tetrahedra

The volume is Π_1 is $\dfrac{1}{6}$ and the volume of Π_2 is $\dfrac{1}{3}$ giving a total of $\dfrac{1}{2}$.

5 Volume of a curved piece

We will compute the volume of the piece bounded by the face of Π_2 with vertices $(1,0,0), (0,0,1), (1,1,1)$ and the hyperboloid $z^2 + x^2 - y^2 = 1$. This face of Π_2 is in the plane $x - y + z = 1$. The projection of this tetrahedral face onto the xy-plane is the triangle with vertices $(0,0),(0,1),(1,1)$ and *the triple integral for the volume of the curved piece is*

$$\boxed{I := \int_0^1 \int_y^1 \int_{1-x+y}^{\sqrt{1+y^2-x^2}} 1 \; dz \; dx \; dy}$$

Carrying out the inner integration we obtain

$$I = \int_0^1 \int_y^1 \sqrt{1+y^2-x^2} - (1-x+y) \; dx \; dy$$

We write

$$I_1 := \int_0^1 \int_y^1 \sqrt{1+y^2-x^2} \; dx \; dy$$

and

$$I_2 := \int_0^1 \int_y^1 (1-x+y) \; dx \; dy.$$

First, using the indefinite integral of $\sqrt{a^2-x^2}$ with $a = 1+y^2$ and then using integration by parts twice we obtain

$$\begin{aligned}
I_1 &= \int_0^1 \left\{ \frac{x}{2}\sqrt{1+y^2-x^2} + \frac{1+y^2}{2} \arcsin\left(\frac{x}{\sqrt{1+y^2}}\right) \right\}_{x=y}^{x=1} dy \\
&= \int_0^1 \left\{ \frac{1+y^2}{2} \arcsin\left(\frac{1}{\sqrt{1+y^2}}\right) - \frac{1+y^2}{2} \arcsin\left(\frac{y}{\sqrt{1+y^2}}\right) \right\} dy \\
&= \frac{\pi}{6} + \frac{1}{2}\int_0^1 \left(y + \frac{y^3}{3}\right) \; dy - \left\{ \frac{\pi}{6} - \frac{1}{2}\int_0^1 \left(y + \frac{y^3}{3}\right) \; dy \right\} \\
&= \int_0^1 \left(y + \frac{y^3}{3}\right) \; dy \\
&= \frac{\ln 2}{3} + \frac{1}{6}
\end{aligned}$$

Moreover, a routine calculation gives

$$I_2 = \frac{1}{3}.$$

Thus the volume of one curved piece is,

$$\begin{aligned} I &= I_1 - I_2 \\ &= \frac{\ln 2}{3} + \frac{1}{6} - \frac{1}{3} \\ &= \frac{\ln 2}{3} - \frac{1}{6} \end{aligned}$$

6 Complete Volume

Therefore the volume of the solid, $\mathbf{R_1}$, in the first quadrant is

$$R_1 = 3I + \frac{1}{2} = 3\left(\frac{\ln 2}{3} - \frac{1}{6}\right) + \frac{1}{2} = \ln 2$$

and the volume of the solid, $\mathbf{R}$, in Archimedes' revenge

$$= 8R_1 = 8\ln 2 = \ln 256.$$

qed!!

7 Final comments

The idea to exploit the symmetry of a component in one octant is due to Li. However, his solution claims that that solid $\mathbf{R_1}$ is composed of ONE tetrahedron of volume $\frac{1}{2}$ and the three congruent curved pieces. We have not been able to follow his reasoning about the tetrahedron. He computes the volume of a curved piece using the following integral whose origin remains a complete mystery to us, namely:

$$I := \frac{1}{2}\int_0^1 \left\{(z^2+1)\left(\frac{\pi}{2} - 2\arctan z\right) + z^2 - 1\right\}\, dz$$

and indeed our attempt to solve that mystery led us to the solution presented here. We welcome any reader's commentary and explanation to clear up of our lack of understanding.

8 Acknowledgements

We thank Oliver Knill for helpful comments . We also acknowledge support from the Vicerrectoría de Investigación of the Universidad de Costa Rica.

References

[1] Archimedes *The Works of Archimedes: The Method*. Edited by T.L. Heath Dover Publications, New York, 1960