8/3/2017 PRACTICE EXAM II **Maths 21a, O. Knill, Summer 2017**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-2 do not require any justifications. Problem 3 only 1-2 words. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

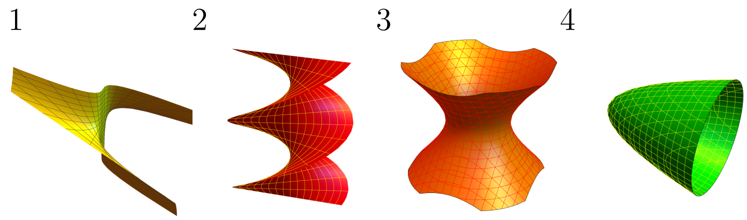
Problem 1) (20 points) No justifications are necessary

1)	T	F	The vectors $\vec{v} = \langle 10, 0, 0\rangle$ and $\vec{w} = \langle 20, 0, 0\rangle$ are perpendicular.
2)	T	F	If f has a maximum at $(0,0)$ under the constraint $g(x,y) = c$, and $\nabla f, \nabla g$ are both not zero at $(0,0)$ then the angle between $\nabla f(0,0)$ and $\nabla g(0,0)$ is zero or π.
3)	T	F	The linearization of $f(x,y) = x^2 + y^3 - x$ at $(2,1)$ is $L(x,y) = 4 + 4(x-2) + 3(y-1)$.
4)	T	F	The flux of the gradient ∇f through a unit sphere $f(x,y,z) = x^2+y^2+z^2 = 1$ is 6 times the volume of the sphere.
5)	T	F	If $\vec{B}(t)$ is the bi-normal vector to a circle contained in the plane $x = 1$, then $\vec{B}(t)$ is always parallel to $\langle 1, 0, 0\rangle$.
6)	T	F	The parametrization $\vec{r}(u,v) = \langle u\cos(v), u, u\sin(v)\rangle$ describes a cone.
7)	T	F	Let E be unit cube with boundary surface S oriented outwards. If $\iint_S \vec{F} \cdot d\vec{S} = 0$, then $\text{div}(\vec{F})(x,y,z) = 0$ everywhere inside E.
8)	T	F	If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.
9)	T	F	If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.
10)	T	F	The volume of the solid E is $\int\int_S \langle x, 2x+z, x-y\rangle \cdot d\vec{S}$, where S is the surface of the solid E oriented outwards.
11)	T	F	The vector field $\vec{F}(x,y,z) = \langle 4x+4y, 4x-4y, z\rangle$ has zero curl and zero divergence everywhere.
12)	T	F	If $\vec{F} = \langle x^2+y, x+y, x-y^2+z\rangle$, then the flux of the vector field $\text{curl}(\text{curl}(F))$ through a sphere $x^2+y^2+z^2 = 1$ is zero.
13)	T	F	If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.
14)	T	F	The equation $\text{div}(\text{grad}(f)) = 0$ is an example of a partial differential equation for an unknown function $f(x,y,z)$.
15)	T	F	The vector $(\vec{i}-2\vec{j}) \times (\vec{i}+2\vec{j})$ is the zero vector if $\vec{i} = \langle 1,0,0\rangle$ and $\vec{j} = \langle 0,1,0\rangle$.
16)	T	F	Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P,L) \geq d(P,\Sigma)$.
17)	T	F	The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.
18)	T	F	If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P,Q)$ is minimized.
19)	T	F	The parametrized surface $\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$ is everywhere perpendicular to the vector field $\vec{F}(x,y,z) = \langle x, y, x^2+y^2\rangle$.
20)	T	F	Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F} = \nabla f$. Then $\vec{r}\,'(t) = \vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$	
$\vec{r}(u,v) = \langle u^2 v, u, v\rangle$	
$x^2 - y^2 = 1 - z^2$	
$2 + x - y^2 - z^2 = 0$	



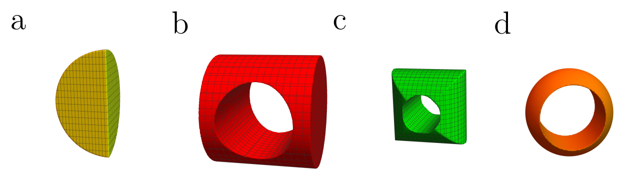
b) (2 points) Match the expressions. There is an exact match.

Integral	Enter A-D
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \; dudv$	
$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dudv$	
$\int_C \lvert\vec{r}\,'(t)\rvert \; dt$	
$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

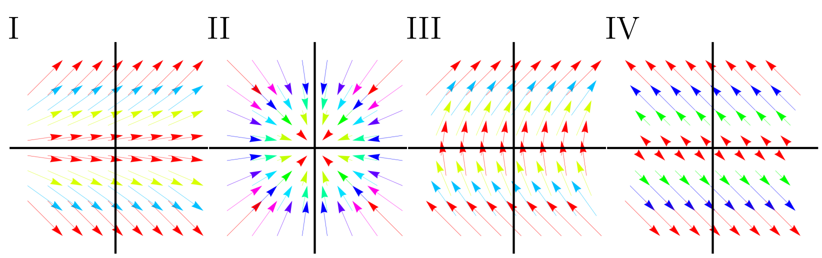
c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$x^2 + z^2 \leq 3, x^2 + z^2 \geq 1, x^2 + y^2 + z^2 \leq 2$	
$x^2 < 8, y^2 + z^2 < 9, x^2 + z^2 > 4$	
$x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0$	
$x^2 + y^2 \leq 4, y^2 + z^2 \leq 4, x^2 + z^2 > 1$	



d) (2 points) The figures display vector fields. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle -x, -y\rangle$	
$\vec{F}(x,y) = \langle 1, y\rangle$	
$\vec{F}(x,y) = \langle y, 1\rangle$	
$\vec{F}(x,y) = \langle -y, y\rangle$	



e) (2 points) Match the partial differential equations:

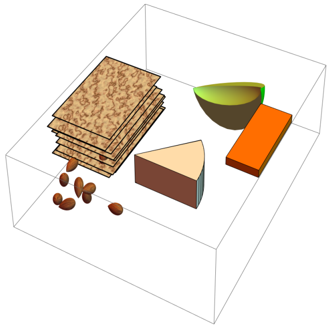
Equation	1-3
Laplace	
Black-Scholes	
Wave	

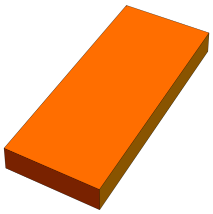
	Partial differential equations
1	$u_t = u - xu_x - x^2 u_{xx}$
2	$u_{tt} - u_{xx} = 0$
3	$u_{tt} + u_{xx} = 0$

Super joker! If you write down a correct equation for the **Burgers equation**, you can regain 2 lost points in this problem. Of course, the maximal number of points to be gained in this problem is still 10.

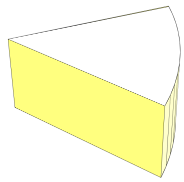
Problem 3) (10 points) No justifications necessary

a) (5 points) Oliver loves **cheese-fruit bistro boxes**. You rarely find any of them in coffee shops, because "Olli the bistro monster" is eating them all (for breakfast, lunch **and** dinner!). One of them contains apples, nuts, cheese and crackers. Lets match the objects and volume integrals:

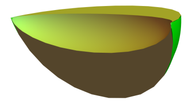




I

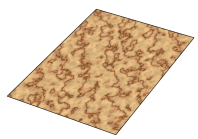


II



III

IV

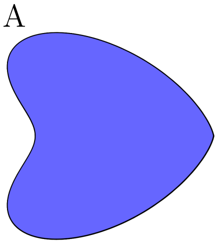


V

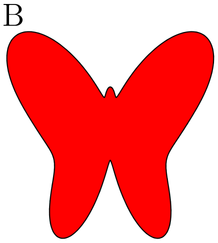
Enter I-V	volume formula
	$\int_1^2 \int_0^\pi \int_0^{\pi/5} \rho^2 \sin(\phi)\, d\theta d\phi d\rho$
	$\int_{-6}^{6} \int_{-10}^{10} \int_{-0.1}^{0.1} 1\, dz dx dy$
	$\int_0^{\pi/4} \int_0^{10} \int_{-5}^{5} r\, dz dr d\theta$
	$\int_{-1}^{1} \int_{-4}^{4} \int_{-8}^{8} 1\, dx dy dz$
	$\int_0^{\pi} \int_0^{2\pi} \int_0^{\cos(2\phi)/4} \rho^2 \sin(\phi)\, d\rho d\theta d\phi$

b) (5 points) Biologist **Piet Gielis** once patented polar regions because they can be used to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when matching the following polar regions:

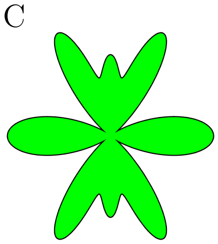
A



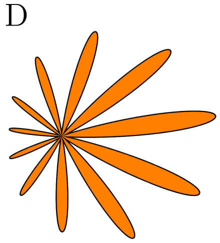
B



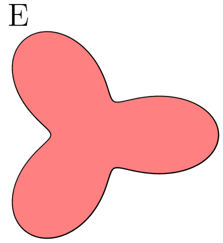
C



D



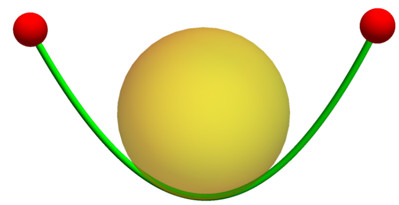
E



Enter A-E	polar region
	$r(t) \leq \lvert 2 + \cos(3t) \rvert$
	$r(t) \leq \lvert \cos(5t) - 5\cos(t) \rvert$
	$r(t) \leq \lvert 1 + \cos(t)\cos(7t) \rvert$
	$r(t) \leq \lvert 8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t) \rvert$
	$r(t) \leq \lvert \sin(11t) + \cos(t)/2 \rvert$

Problem 4) (10 points)

A parabola is parametrized by $\vec{r}(t) = \langle t, t^2, 0 \rangle$, where $-1 \leq t \leq 1$. We are interested in some properties at the tip $\vec{r}(0) = \langle 0, 0, 0 \rangle$.



a) (2 points) Find the speed $|\vec{r}\,'(t)|$ at $t = 0$.

b) (2 points) Find the acceleration vector $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Find the curvature κ at $t = 0$.

d) (3 points) Write down the arc length integral of the curve. We have evaluated that in class. No need to evaluate it here.

Problem 5) (10 points)

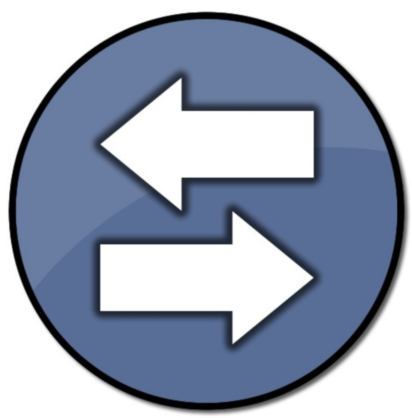
Some chain rule related stuff:

a) (4 points) Estimate $1.002^6 \cdot 0.998^3$ using linear approximation.

b) (3 points) Find the tangent plane to $x^2 + y^4 + z^6 = 3$ at $(1, 1, 1)$.

c) (3 points) Find $D_{\vec{v}} f(1, 1)$ for $\vec{v} = \frac{\langle 5,12 \rangle}{13}$ and $f(x, y) = x^{13} + y^{13}$.

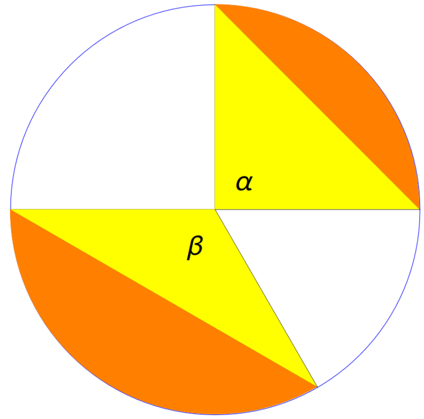
Problem 6) (10 points)



Evaluate the following triple integral computing a volume in cylindrical coordinates:

$$\int_0^{\pi^2} \int_{\sqrt{r}}^{\pi} \int_0^{\sin(\theta)/(r\theta^2)} r \; dz d\theta dr \; .$$

Problem 7) (10 points)



The difference between the total area of two triangles and the area of the two sector regions attached to them in a disc of radius 2 is

$$f(\alpha, \beta) = 2\sin(\alpha) - \alpha + 2\sin(\beta) - \beta \; .$$

a) (5 points) Find the maximum of f using the second derivative test.

b) (5 points) Use Lagrange to find the maximum of f under the constraint $g(\alpha, \beta) = \alpha + \beta = \pi/2$.

Problem 8) (10 points)

This problem was written while watching the space opera **"Jupiter ascending"**, with a plot so predictable that it is a perfect movie to do some math on the side:

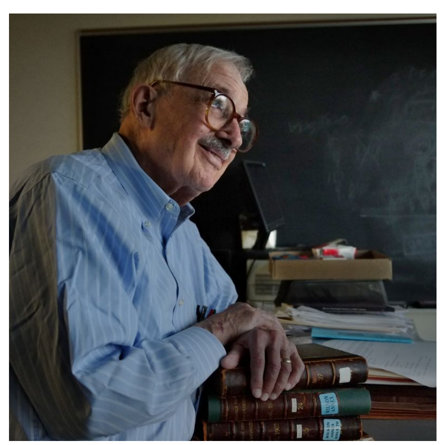
a) (5 points) The genetically engineered soldier Channing Tatum surfs with velocity

$$\vec{v}(t) = \langle \sin(t), \sin(2t), \cos(t) \rangle \; .$$

He starts at the origin. Where has he surfed to at time $t = 2\pi$?

b) (5 points) Find a parametrization of the tangent line to the curve at $t = 2\pi$.

Problem 9) (10 points)



About 40 years ago, economists like **John Chipman** (who taught at Harvard) defined consumer surplus as a line integral $\int_C \vec{F}(r(t)) \cdot r'(t) dt$, where $C : \vec{r}(t)$ is a path in a prize xy-plane. The vector field $\vec{F} = (P, Q)$ consists of demand functions P, Q. Find the consumer surplus for

$$\vec{r}(t) = \langle t, t^2 \rangle$$

and prize field

$$\vec{F}(x, y) = \langle \sin(x) - y\sin(xy), \cos(y) - x\sin(xy)$$

from $t = 0$ to π.

Remark: Books written by economists are jargon and math heavy. You read in one of these books that "the constancy of the marginal utility of income utilizes Hicks-Slutsky partial differential equations." Having taken Math21a, and looking at this PDE, you understand what it implies: "$\vec{F}$ is conservative."

Problem 10) (10 points)

A **band aide strip** has the shape of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos^3(t), \sin(t) + \cos(t) \rangle \ ,$$

where $0 \leq t \leq 2\pi$. Find its area!

Problem 11) (10 points)

Der **Titan Aerospace Solara 50** is a drone with 150 feet wing span being able to carry a payload of 250 pounds possibly for 5 years. Google bought the company and aims to use it to bring internet to new regions. Assume Solara flies along a closed loop C:

$$\vec{r}(t) = \langle 10\cos(t), 10\sin(t), 100 \rangle$$

with $0 \leq t \leq 2\pi$ under the influence of the wind force

$$\vec{F}(x, y, z) = \langle \sin(x^3) - 3y, 2x + \cos(y^5), \sin(z^2) + z^{11} \rangle \ .$$

How much work $\int_C \vec{F} \cdot \vec{dr}$ does it have to do?

Problem 12) (10 points)

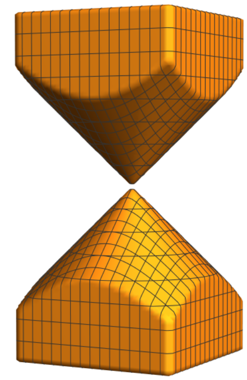
We build a new **nozzle** for a 3D printer. It has the shape of the solid region obtained by intersecting the solid cone

$$x^2 + y^2 \leq z^2$$

with the cuboid

$$-1 \leq x \leq 1, -1 \leq y \leq 1, -2 \leq z \leq 2 \ .$$

Find its volume.



Problem 13) (10 points)

On July 31, 2015, a **blimp** flew over the science center. It was a windy day and the air ship E with ellipsoid surface hull S parametrized by

$$\vec{r}(u,v) = \langle 10\cos(v)\sin(u), 3\sin(v)\sin(u), 2\cos(u)\rangle$$

and volume $V(E) = 80\pi$ had to fight a rather turbulent wind velocity field

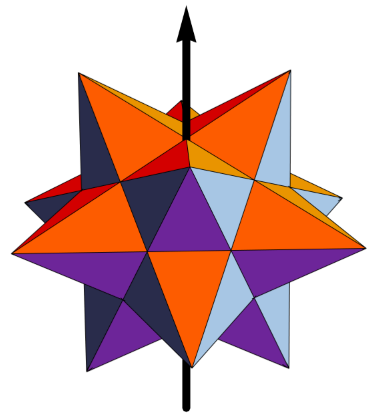
$$\vec{F}(x,y,z) = \langle x + \cos(y^2 z), y + \sin(xz^2), z + \sin(x^2 y) \ .$$

The flux of $\vec{F}$ through S is the total force acting onto the ship. Find that flux, assuming that the surface S is oriented outwards.

Photo: Oliver, July 31, 2015.

Problem 14) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \; dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle y^5 + x, z^5 + y, x^5 + z \rangle$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = \langle x^3/3, y^3/3, 0 \rangle$$

through S?