

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
| 5      |  | 10  |
| 6      |  | 10  |
| 7      |  | 10  |
| 8      |  | 10  |
| 9      |  | 10  |
| 10     |  | 10  |
| 11     |  | 10  |
| 12     |  | 10  |
| 13     |  | 10  |
| Total: |  | 140 |

Problem 1) (20 points) No justifications are necessary

- 1) ☒ T ☐ F      The equation  $f_{xx} = f_y$  is the heat equation, an example of an partial differential equation.

**Solution:**

Yes, we usually use other letters, but it is true

- 2) ☒ T ☐ F      The triple scalar product between three vectors in a plane is always zero.

**Solution:**

It is the signed volume of the parallel epiped.

- 3) ☐ T ☒ F      The length of the gradient  $|\nabla f|$  is maximal at a maximum of  $f$ .

**Solution:**

It is zero at a maximum.

- 4) ☒ T ☐ F      Given a line  $L$  inside a plane  $\Sigma$  and a point  $P$ . The distance between  $L$  and  $P$  is larger or equal than the distance from  $\Sigma$  and  $P$ .

**Solution:**

Yes, restricting to a line can make the distance only larger or be kept the same.

- 5) ☐ T ☒ F      For a curve on the sphere, the  $\vec{B}$  vector in the TNB frame points always outwards.

**Solution:**

It depends on the curve. On a circle circling the north pole counter clockwise the  $B$  vector points outwards. Near the south pole inwards.

- 6) ☐ T ☒ F      The surface  $x^2 - y^2 + z^2 = -1$  is a one sheeted hyperboloid.

**Solution:**

It is a two sheeted hyperboloid.

- 7) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The equation  $u_{tt} = uu_x + u_{xx}$  is called the Burger's equation.

**Solution:**

There is only  $u_t = \dots$

- 8) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The fundamental theorem of line integrals assures that the line integral of any gradient field along any path zero.

**Solution:**

It needs a closed path.

- 9) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The dot product between  $\vec{v} = \langle 1, 1 \rangle$  and  $\vec{w} = \langle 1, 0 \rangle$  is is equal to the cross product  $\vec{v} \times \vec{w}$  between these two vectors.

**Solution:**

Compute both, and it is almost true. In two dimensions, the cross product is also a scalar. But the sign is off.

- 10) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 If  $\vec{T}(t)$  is the unit tangent vector. Then  $\vec{T}' = \vec{0}$  because the length of  $\vec{T}$  stays the same.

**Solution:**

The length staying the same is not the same than the vector staying the same

- 11) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 We can look at the discriminant  $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$  at every point  $(x, y)$  If  $D$  has a maximum at  $(0, 0)$ , then the gradient of  $D$  is zero at  $(0, 0)$ .

**Solution:**

Why not? It is true for all functions. So also for  $D$ .

- 12) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The curl of a conservative field is zero. [PS: this is the reason why all republicans have straight hair!]

- 13) 

|   |   |
|---|---|
| T | F |
|---|---|

 The flux of the curl of  $\vec{F}$  through the surface  $S$  is positive, then  $S$  has a boundary.

**Solution:**

By Stokes theorem

- 14) 

|   |   |
|---|---|
| T | F |
|---|---|

 $\vec{r}(u, v) = \langle u^3, 0, v^3 \rangle$  parametrizes a plane.

**Solution:**

Indeed. It is the same than if  $u^3, v^3$  are replaced by  $u, v$ .

- 15) 

|   |   |
|---|---|
| T | F |
|---|---|

 If the integral  $\int \int \int_G \operatorname{div}(\vec{F}(x, y, z)) \, dx dy dz$  is zero for all balls  $G = \{x^2 + y^2 + z^2 \leq r\}$ , then the divergence is zero.

**Solution:**

Take  $F = \langle x^3, 0, 0 \rangle$

- 16) 

|   |   |
|---|---|
| T | F |
|---|---|

 Assume  $\vec{r}(t)$  is a stream line of a vector field  $F = \nabla f$ . Then the velocity of  $\vec{r}$  is zero if we are at a critical point of  $f$ .

**Solution:**

By definition  $\vec{r}'(t) = F(\vec{r}(t)) = \nabla f(\vec{r}(t))$ .

- 17) 

|   |   |
|---|---|
| T | F |
|---|---|

 If  $\vec{F} = \operatorname{curl}(\vec{G})$  and  $\operatorname{div}(\vec{F}) = 0$ , then  $\operatorname{div}(\vec{G}) = 0$ .

**Solution:**

$\operatorname{div}(\vec{F}) = 0$  is always true in that case.

- 18) 

|   |   |
|---|---|
| T | F |
|---|---|

 It is possible that  $\vec{v} \cdot \vec{w} > 0$  and  $\vec{v} \times \vec{w} = \vec{0}$ .

**Solution:**

Take  $\vec{v} = \vec{w}$ .

- 19) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The directional derivative  $D_{\vec{v}}(r)$  of a curve  $\vec{r}(t)$  is defined as  $\vec{v} \cdot \vec{r}'(t)$ .

**Solution:**

We have not defined directional derivative for a curve

- 20) 

|   |
|---|
| T |
|---|

|   |
|---|
| F |
|---|

 The vector projection  $P_{\vec{w}}(\vec{v})$  defines a product which is commutative:  
 $P_{\vec{w}}(\vec{v}) = P_{\vec{v}}(\vec{w})$ .

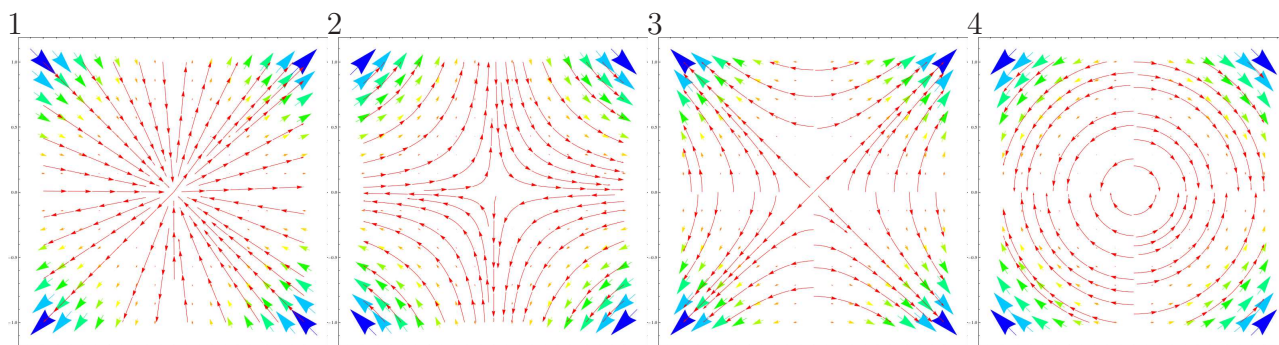
**Solution:**

$P_{\vec{w}}(\vec{v})$  points in the direction of  $\vec{w}$  while  $P_{\vec{v}}(\vec{w})$  points in the direction of  $\vec{v}$ .

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the following vector fields

| Field                                         | Enter 1-4 |
|-----------------------------------------------|-----------|
| $\vec{F}(x, y) = \langle x^2y, y^2x \rangle$  |           |
| $\vec{F}(x, y) = \langle x^2y, -y^2x \rangle$ |           |
| $\vec{F}(x, y) = \langle xy^2, x^2y \rangle$  |           |
| $\vec{F}(x, y) = \langle xy^2, -x^2y \rangle$ |           |



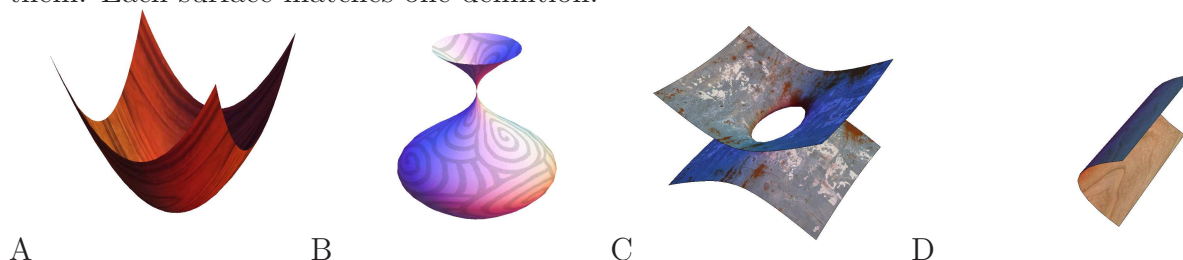
b) (2 points) Match the following names of partial differential equations with functions  $u(t, x)$  which satisfy the differential equation and with formulas defining these equations.

| equation  | A-D | 1-4 |
|-----------|-----|-----|
| Laplace   |     |     |
| heat      |     |     |
| wave      |     |     |
| transport |     |     |

|   |                         |
|---|-------------------------|
| A | $u(t, x) = t^2 + x^2$   |
| B | $u(t, x) = t^2 - x^2$   |
| C | $u(t, x) = \sin(x + t)$ |
| D | $u(t, x) = x^2 + 2t$    |

|   |                                |
|---|--------------------------------|
| 1 | $u_t(t, x) = u_{xx}(t, x)$     |
| 2 | $u_t(t, x) = u_x(t, x)$        |
| 3 | $u_{tt}(t, x) = -u_{xx}(t, x)$ |
| 4 | $u_{tt}(t, x) = u_{xx}(t, x)$  |

c) (4 points) The following surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.



| Enter A-D here | Function or parametrization                                                       |
|----------------|-----------------------------------------------------------------------------------|
|                | $x - 4z^2 = 1$                                                                    |
|                | $\vec{r}(u, v) = \langle (1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v), u \rangle$ |
|                | $\vec{r}(u, v) = \langle u, v, u^2 + v^2 \rangle$                                 |
|                | $4x^2 + y^2 - 9z^2 = 1$                                                           |

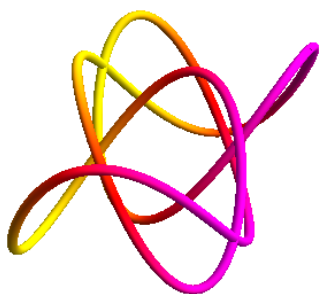
**Solution:**

- a) 1,2,3,4  
b) 3,1,4,2 and B,D,A,C  
c) D,B,A,C

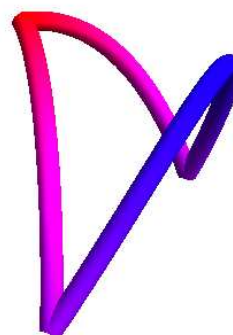
Problem 3) (10 points) No justifications are necessary

- a) (3 points) Closed curves in space are called knots. Match them

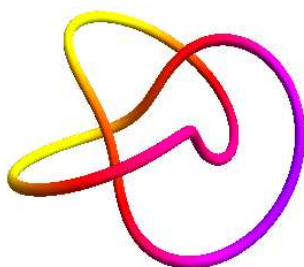
| Formula                                                                                                               | Enter A,B,C,D |
|-----------------------------------------------------------------------------------------------------------------------|---------------|
| $\vec{r}(t) = \langle  \cos(t) ,  \sin(t) + \cos(t) ,  \cos(2t)  \rangle$                                             |               |
| $\vec{r}(t) = \langle \cos(3t), \cos(t) + \sin(3t), \cos(5t) \rangle$                                                 |               |
| $\vec{r}(t) = \langle (2 + \cos(\frac{3t}{2})) \cos(t), (2 + \cos(\frac{3t}{2})) \sin(t), \sin(\frac{3t}{2}) \rangle$ |               |
| $\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$                                                              |               |



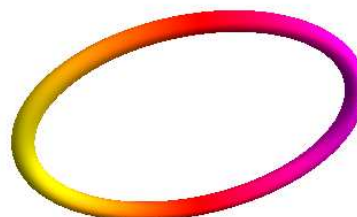
A



B



C



D

- b) (5 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

| Theorem                        | Grad | Curl | Div | Line integral | Flux integral |
|--------------------------------|------|------|-----|---------------|---------------|
| Fundamental Thm line integrals |      |      |     |               |               |
| Greens theorem                 |      |      |     |               |               |
| Divergence theorem             |      |      |     |               |               |
| Stokes' theorem                |      |      |     |               |               |
| Second derivative test         |      |      |     |               |               |

**Solution:**

c) (2 points) Which of the following formulas give the unit tangent vector?

| label | formula                       |
|-------|-------------------------------|
| A     | $\int_a^b  \vec{r}'(t)  dt$   |
| B     | $\vec{r}'(t)/ \vec{r}'(t) $   |
| C     | $\vec{T}'(t)/ \vec{r}'(t) ^3$ |
| D     | $ \vec{T}'(t) / \vec{r}'(t) $ |

**Solution:**

a) B,A,C,D

b) Fundamental: Grad, Line integral

Green: Curl, Line integral

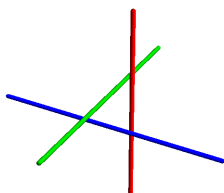
Divergence: Divergence, Flux integral

Stokes: Curl, Line integral, Flux integral

Second derivative test: no match.

c) B

Problem 4) (10 points)



a) (4 points) Find a parametrization  $\vec{r}(t) = P + t\vec{v}$  of the line defined by  $x = 1, y = 1$ .

b) (6 points) Find all the three distances between the line

$$x = 1, y = 1,$$

the line

$$x = -1, z = 1$$

and the line

$$y = -1, z = -1.$$

**Solution:**

a)  $\vec{r}_1(t) = \langle 1, 1, t \rangle$ . The other lines are  $\vec{r}_2(0) = \langle -1, t, 1 \rangle$  and  $\vec{r}_3(0) = \langle t, -1, -1 \rangle$ .

c) To get the distances, use the standard distance formulas using the points  $P = \vec{r}_1(0) = \langle 1, 1, 0 \rangle$ ,  $Q = \vec{r}_2(0) = \langle -1, 0, 1 \rangle$  and  $R = \vec{r}_3(0) = \langle 0, -1, -1 \rangle$ . We get that all three distances are 2.



Problem 5) (10 points)



Estimate

$$\left(\frac{10001 \cdot 9999}{10002}\right)^{1/4}$$

Magic!

Pic-

ture credit: (<http://chibionpu.webs.com>)

**Solution:**

The function to consider is  $f(x, y, z) = xy/z$ . We evaluate at  $x + 0 = y_0, z - 0 = 10000$ , where  $f$  is equal to 10. We have  $f_x(x_0, y_0, z_0) = 1/4000$ ,  $f_y(x_0, y_0, z_0) = 1/4000$  and  $f_z(x_0, y_0, z_0) = -1/4000$ . The estimate is  $10 + 1/4000 - 1/4000 - 2/4000 = 10 - 1/2000$ . The answer is 10-1/2000.

Problem 6) (10 points)



Find and classify the minima and maxima of the function

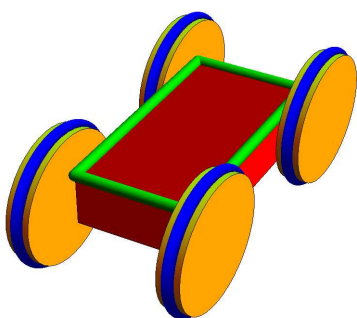
$$f(x, y) = 2x^3 - 15x^2 + 36x - y^2 - 2y.$$

The function is remarkable and special because it does not have any interpretation whatsoever. It is therefore a "very special function" and dear to us. Determine also whether  $f$  has a global maximum somewhere.

**Solution:**

| x | y  | D   | $f_{xx}$ | type    | f  |
|---|----|-----|----------|---------|----|
| 2 | -1 | 12  | -6       | maximum | 29 |
| 3 | -1 | -12 | 6        | saddle  | 28 |

Problem 7) (10 points)



We build a toy car. Minimize the weight

$$f(x, y) = x^2 + 4\pi y^2$$

of the car with length=width  $x$  and wheel radius  $y$  under the constraint that the combined steel frame length and tire length

$$g(x, y) = 4x + 4\pi y = 8$$

is constant.

**Solution:**

The Lagrange equations are

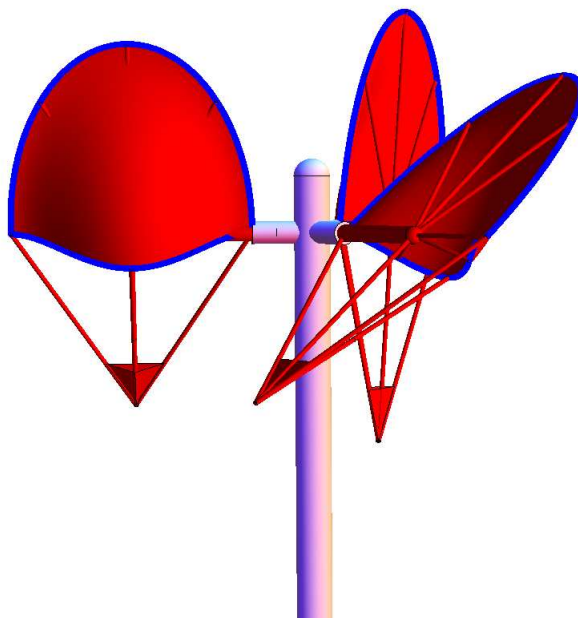
$$2x = \lambda 4$$

$$8\pi y = \lambda 4\pi$$

$$4x + 4\pi y = 8$$

Eliminating  $\lambda$  from the first two equations gives  $x = 4y$ . Plugging this into the third and solving gives  $x = 8/(4 + \pi), y = 2/(4 + \pi)$

Problem 8) (10 points)



**Porter square** in Cambridge features a moving art project. It proves that paddle wheels and curl are omni present. To build that model for google earth, we idealized one of the blades as the surface

$$\vec{r}(t, s) = \langle 2 \cos(t) \sin(s), 4 \sin(t) \sin(s), \cos^4(s) \rangle$$

with  $t, s \in [0, \pi]$ . The surface  $S$  is oriented so that the boundary consists of two parts. The surface boundary can be parametrized as

$$\vec{r}_1(t) = \langle 2 \sin(t), 0, \cos^4(t) \rangle, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

and

$$\vec{r}_2(t) = \langle 2 \cos(t), 4 \sin(t), 0 \rangle, 0 \leq t \leq \pi.$$

The wind force is given by the curl of the vector field  $\vec{F}(x, y, z) = \langle y, z, 0 \rangle$ . Find the flux of  $\text{curl}(\vec{F})$  through the surface  $S$ .

**Solution:**

Use Stokes theorem and compute two line integrals.

$$\int_{-\pi/2}^{\pi/2} \langle 0, \cos^4(t), 0 \rangle \cdot \langle 2 \cos(t), 0, 4 \cos^3(t) \rangle dt = 0$$

$$\int_0^\pi \langle 4 \sin(t), 0, 0 \rangle \cdot \langle -2 \sin(t), 4 \cos(t), 0 \rangle dt = -8 \int_0^\pi \sin^2(t) dt = -4\pi.$$

The result is the sum of these two line integrals which is  $\boxed{-4\pi}$ .



Image: Pforzheim Jewelry Museum.

August Kiehle, 1885

A chain has the shape:

$$\vec{r}(t) = \langle 2t, 18 + e^t + e^{-t}, 0 \rangle .$$

a) (6 points) What is the length of this chain parametrized for  $-1 \leq t \leq 1$ .

b) (4 points) You are so excited, you drop the chain. The center piece which is initially at  $\vec{r}(0) = \langle 0, 0, 20 \rangle$  will fall on the  $z$  axes while subject to the gravitational force  $\vec{r}''(t) = \langle 0, 0, -10 \rangle$  and having zero velocity  $\vec{r}'(0) = \langle 0, 0, 0 \rangle$  initially. How long does it take to hit the ground?

### Solution:

a)  $\vec{r}'(t) = \langle 2, e^t - e^{-t} \rangle$  which has length  $|\vec{r}'(t)| = \sqrt{4 + e^{2t} - 2 + e^{-2t}} = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t}$ . Integrating this from  $-1$  to  $1$  gives  $2e + 2e^{-1}$ .

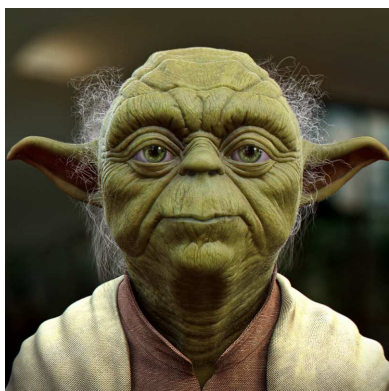
b)  $\vec{r}''(t) = \langle 0, 0, -10 \rangle$ .

$\vec{r}'(t) = \langle 0, 0, -10t \rangle$ .

$\text{vecr}(t) = \langle 0, 0, 20 - 5t^2 \rangle$ .

Which is at the ground for  $t = 2$ .

Problem 10) (10 points)



Evaluate the following integral

$$\int_0^4 \int_{\sqrt{y/4}}^1 \int_0^{e^{-x^4}} x \, dz \, dx \, dy .$$

Remember what Yoda said.

**Solution:**

Start integrating:

$$\int_0^4 \int_{\sqrt{y/4}}^1 e^{-x^4} x \, dx dy .$$

Now we have a double integral which we can not solve. Yoda tells "the orientation change you must":

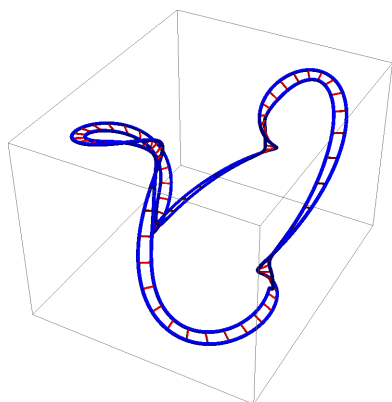
$$\int_0^1 \int_0^{4x^2} e^{-x^4} x \, dy dx .$$

Now we can integrate

$$\int_0^1 e^{-x^4} 4x^3 \, dx = -e^{-x^4} \Big|_0^1 = 1 - e^{-1} .$$

The answer is  $\boxed{1 - e^{-1}}$ .

Problem 11) (10 points)



In 2006, Oliver submitted a roller coaster suggestion to a Harvard sustainability competition. Imagine to ride from the science center to Allston in a minute with some loop over the Charles. Together with a coffee shop on top of the science center and hot dog and coffee stands near the river and Storrow drive under ground, Cambridge would become the funnest city in the world. So far, it is only a dream. Assume the roller coaster is the path

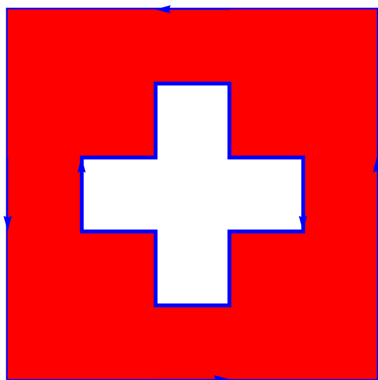
$$\vec{r}(t) = \langle 4 \cos(t) + \sin(5t), 4 \sin(t) - 2 \cos(5t), 4 \sin(2t) \rangle$$

and assume, the car experiences a force  $\vec{F}(x, y, z) = \langle x^{100}, y, z^{20} \rangle$ . What work is done to go from  $\vec{r}(0)$  to  $\vec{r}(\pi)$ ?

**Solution:**

Because the curl of  $\vec{F}$  is zero, the field is a gradient field and we deal with a fundamental theorem of line integral problem. We find the potential  $f(x, y, z) = x^{101}/101 + y^2/2 + z^{21}/21$ . Now evaluate it at  $\vec{r}(0) = \langle 4, -2, 0 \rangle$  and  $\vec{r}(\pi) = \langle -4, 2, 0 \rangle$  and get  $f(\vec{r}(\pi)) - f(\vec{r}(0)) = \boxed{-2 \cdot 4^{101}/101}$ .

Problem 12) (10 points)



Let  $G$  be the red complement of the cross in the Swiss flag. The entire flag has dimension  $5 \times 5$  and the cross consists of 5 squares of unit length. Let  $C$  be the boundary of the red complement region oriented so that the region is to the left. The boundary consists of two curves. Find the line integral

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) ,$$

where

$$\vec{F}(x, y) = \langle x^9 y^{10} + 7y, y^9 x^{10} - 3x + \sin(y) \rangle .$$

### Solution:

By Greens theorem, the line integral along the boundary is equal to

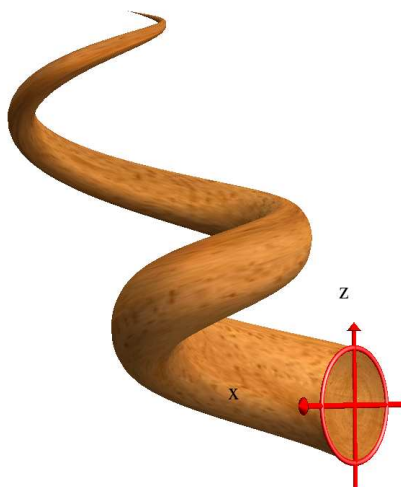
$$\int \int_G \text{curl}(\vec{F}) \, dxdy .$$

The curl is  $\text{curl}(\vec{F}) = Q_x - P_y = -3 - 7 = -10$ . Therefore

$$\int \int_G \text{curl}(\vec{F}) \, dxdy = -10 \int \int_G 1 \, dxdy = -10 \text{Area}(G) .$$

The area of the red region is  $25 - 5 = 20$ . The answer is  $\boxed{-200}$ .

Problem 13) (10 points)



A twisted Swiss Alphorn  $S$  is parametrized as

$$\vec{r}(u, v) = \left\langle 4\pi\left(4 + \left(2 - \frac{u}{2\pi}\right)\cos(v)\right)\cos(u) - u, 4\pi\left(4 - \frac{u}{2\pi}\cos(v)\right)\sin(u), 4\pi\left(2 - \frac{u}{2\pi}\right)\sin(v) + u \right\rangle,$$

where  $0 \leq u \leq 4\pi$  and  $0 \leq v \leq 2\pi$ . Together with a disc  $D$  obtained in the  $xz$ -plane  $y = 0$ , the closed surface  $S \cup D$  bounds the solid  $E$ , oriented outwards. You are given that the volume of  $E$  is the same than the volume of a cone  $x^2 + y^2 \leq (4\pi - z)^2, 0 \leq z \leq 4\pi$ . The vector field

$$\vec{F}(x, y, z) = \langle -y + x, x + z^2, z + x^2 \rangle$$

is the sound energy transfer field. Find the flux of  $\vec{F}$  through  $S \cup D$ .

**Solution:**

Since the surface is closed, this is a divergence theorem problem. The flux is the integral of the integral

$$\int \int \int_E \operatorname{div}(\vec{F}) \, dx dy dz = 2\operatorname{Vol}(E)$$

Since the volume was given  $(4\pi)^2\pi(4\pi)/3$  (a cone has volume base area times height divided by 3), the answer is  $128\pi^4/3$ .

Remark. There is a justification of why "straighening the alphorn" does not change the volume: a generalization of the Pappus centroid theorem tells that the volume of a tube of radius  $r$  about a curve is the length times  $\pi r^2$  if the curve is straight. This has been generalized and became a "theory of tubes", a powerful tool in Riemannian geometry. But intuitively, it can be seen by building a horn first as a cone, then bend it in the given shape. The bending does not change the mass, and to a good approximation does not change the volume.