7/6/2017 FIRST HOURLY **Maths 21a, O.Knill, Summer 2017**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-4 and 9, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The Cauchy-Schwartz inequality assures that $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$ for all $\vec{v}, \vec{w}$.

Solution:
Yes, this is the right inequality

2) T F The two planes $3x + 4y + z = 4$ and $6x + 8y + 2z = 4$ intersect in a line.

Solution:
They are parallel.

3) T F If the distance between two points P and Q is zero, then $P = Q$.

Solution:
If P and Q were different, then the points would be different.

4) T F The symmetric equations of $\vec{r}(t) = \langle 2t, 2t, t \rangle$ is $x = y = 2z$.

Solution:
You can check by plugging in $x = 2t, y = 2t, z = t$.

5) T F The arc length of a circle with constant curvature r is $2\pi r$.

Solution:
It is $2\pi/r$

6) T F The surface $x^2 - y^2 + 4y = z^2$ is a two sheeted hyperboloid.

Solution:
Complete the square to get $x^2 - (y-2)^2 - z^2 = -4$ is one sheeted hyperboloid.

7) | T | **F** | There are two vectors $\vec{v}, \vec{w}$ in space for which the angle between the vectors is $3\pi/2$.

Solution:
We have defined the vector to be a number in $[0\pi]$.

8) | T | **F** | If the acceleration of a curve $\vec{r}(t)$ is zero at all times, then $\vec{r}(t) = \vec{r}(0)$.

Solution:
The velocity can be nonzero. The parametrization $\vec{r}(t) = \langle t, t, t \rangle$ for example has zero acceleration at all times t.

9) | **T** | F | The lines $\vec{r}(t) = \langle 3t, 4t, 5t \rangle$ and $\vec{r}(t) = \langle -4t, 3t, 0 \rangle$ intersect perpendicularly.

Solution:
The velocity vectors are perpendicular at the intersection point $(0, 0, 0)$.

10) | **T** | F | The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axes.

Solution:
It is the south pole.

11) | T | **F** | Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \cdot \vec{v})||\vec{w}| = |\vec{u}||\vec{v} \cdot \vec{w}|$.

Solution:
Take $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \vec{w} = \langle 0, 1, 0 \rangle$.

12) | T | **F** | The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a plane.

Solution:
Multiply both by ρ to get $z = \rho^2$ which is a sphere.

13) | **T** | F | The arc length of the curve $\langle \sin(t), \cos(t), t \rangle$ from $t = 0$ to $t = 2\pi$ is larger than 2π.

Solution:
It is a helix whose projection is a circle.

14) | **T** | F | The surface parametrized by $\vec{r}(u, v) = \langle v\sin(u), v\cos(u), 0 \rangle$ is a plane.

Solution:
Indeed it is the plane $z = 0$.

15) | **T** | F | The length of the cross product of two non-parallel vectors $\vec{v}$ and $2\vec{w}$ is larger than the length of the cross product of the vectors $\vec{v}$ and $\vec{w}$.

Solution:
It is non-zero. The second is twice as large.

16) | T | **F** | It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

Solution:
One is bounded, the other not.

17) | **T** | F | The function $f(x, y) = x^4y^4/(x^2 + y^2)$ is continuous at $(0, 0)$ in the sense we have defined continuity: there is a value one can assign to $f(0, 0)$ so that the function is continuous everywhere.

Solution:
Use Polar coordinates

18) T F Given two curves $\vec{r}(t)$ and $\vec{s}(t)$ for which $\vec{r}\,''(t) = \vec{s}\,''(t)$ for all t and $\vec{r}(0) = \vec{s}(0)$, then $\vec{r}(t) = \vec{s}(t)$.

Solution:
We need also the velocities to match.

19) T F If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$, then $\vec{v} = \vec{w}$ or $\vec{v} = -\vec{w}$.

Solution:
It can be the one is zero. It also can be that $\vec{v}$ and $\vec{w}$ are parallel without being the same.

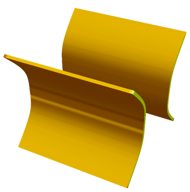
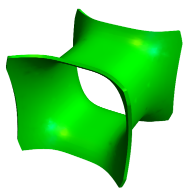
20) T F The point $M = (A+B+C)/3$ is equidistant from A and B and C meaning: $d(A, M) = d(B, M) = d(C, M)$.

Solution:
Take for example $A = (-1, 0, 0), B = (0, 0, 0), C = (1, 0, 0)$.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.


I

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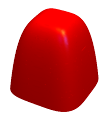
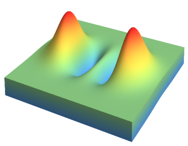

III

Function $g(x, y, z) =$	Enter O,I,II or III
$x^4 + z^4 + z$	
$x^4 - y^2 + z^2$	
$x - y$	
$x^4 - y^4$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.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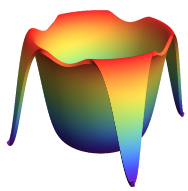
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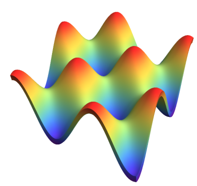
III

Function $f(x, y) =$	Enter O,I,II or III
$\sin(x)\sin(y)$	
$\sin(x^2 + y^2 - 3)$	
$x\cos(y^2)$	
$\exp(-x^2 - y^2)(x^4 - x^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.


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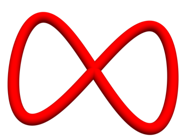
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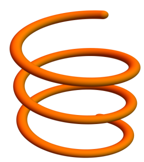
III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t^2/5, \sin(4t), \cos(4t) \rangle$	
$\langle \cos(t), \sin(2t), \cos(t) \rangle$	
$\langle t, t, t \rangle$	
$\langle \sin(t), \cos(t), \sin(3t) \rangle$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

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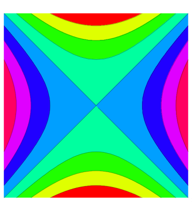
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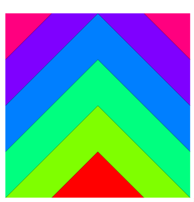
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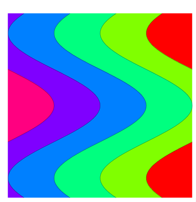
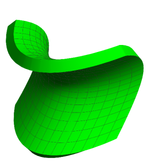


Function $g(x, y) =$	Enter O, I,II or III
$\sin(x^2 - y^2)$	
$\sin(x) - y$	
$\|x\| + y$	
$10x^2 + y^2$	

e) (2 points) Match the quadrics. Enter O if there is no match.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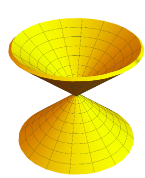
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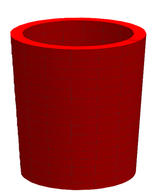

III

Quadric	Enter O,I,II or III
$y = x^2 - z^2$	
$x^2 + y^2 = 1$	
$x^2 + y^2 + z^2 = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

Solution:
a) 3,2,0,1
b) 3,2,0,1
c) 1,2,0,3
d) 1,3,2,0
e) 1,3,0,0,2

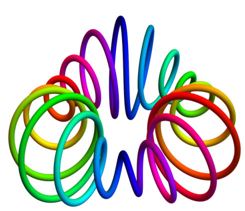
Problem 3) (10 points)	(No justifications are needed in this problem)

a) (4 points) If you combine the left and upper statement you get a statement. Check the ones which are true for any two vectors $\vec{u}$ and $\vec{v}$.

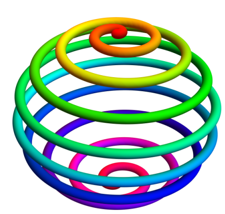
Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0\rangle$	can be nonzero vector
$(\vec{u} \times \vec{v}) \times (\vec{u} \times \vec{v})$				
$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$				
$\text{proj}_{\vec{u}\times\vec{v}}\vec{u}$				
$\text{proj}_{\vec{u}\times\vec{v}}\vec{v}$				

b) (3 points) The three curves parametrized here are each located on a surface. This helps you to match the curves.

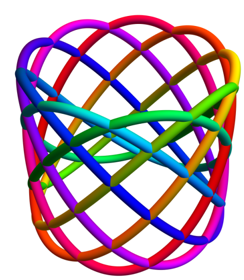
$\vec{r}(t) =$	Enter A-C
$\langle \cos(9t)\sin(t), \sin(9t)\sin(t), \cos(t)\rangle$	
$\langle (5 + (2 + \cos(3t))\sin(9t))\cos(t), (5 + (2 + \cos(3t))\sin(9t))\sin(t), (2 + \cos(3t))\cos(9t)\rangle$	
$\langle \cos(7t), \sin(7t), \sin(9t)\rangle$	



B



A



C

c) (3 points) Complete the "multiplication table". In each case, enter either $\vec{T}, \vec{N}, \vec{B}$ or $\vec{0}$ in each of the 9 boxes

$\text{proj}_{\vec{T}}(\vec{T}) =$		$\text{proj}_{\vec{T}}(\vec{N}) =$		$\text{proj}_{\vec{T}}(\vec{B}) =$	
$\text{proj}_{\vec{N}}(\vec{T}) =$		$\text{proj}_{\vec{N}}(\vec{N}) =$		$\text{proj}_{\vec{N}}(\vec{B}) =$	
$\text{proj}_{\vec{B}}(\vec{T}) =$		$\text{proj}_{\vec{B}}(\vec{N}) =$		$\text{proj}_{\vec{B}}(\vec{B}) =$	

Solution:
a)

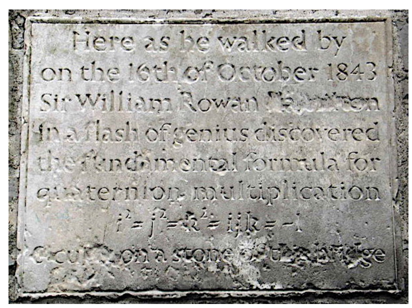
Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0 \rangle$	can be nonzero vector
$(\vec{u} \times \vec{v}) \times (\vec{u} \times \vec{v})$			*	
$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$		*		
$\text{proj}_{\vec{u} \times \vec{v}} \vec{u}$			*	
$\text{proj}_{\vec{u} \times \vec{v}} \vec{v}$			*	

b) ABC c) Everything $\vec{0}$ except in the diagonal, where one has $\vec{T}$, $\vec{N}$,$\vec{B}$.

Problem 4) (10 points)

On October 16 of 1843, **Sir William Hamilton** discovered quaternions. This was also the birth date of the dot and cross product. The reason is that for quaternions without constant part, one has

$$\begin{aligned} (ai + bj + ck)(ui + vj + wk) &= \langle a, b, c \rangle \cdot \langle u, v, w \rangle \\ &+ \langle a, b, c \rangle \times \langle u, v, w \rangle \end{aligned}$$



The quaternion plaque in Dublin shows that the postulate $i^2 = j^2 = k^2 = ijk = -1$ does the trick.

You can ignore the above story and compute some dot and cross products.
We work with

$$\vec{v} = \langle 3, 4, 5 \rangle = 3i + 4j + 5k$$

and

$$\vec{w} = \langle 2, 2, 1 \rangle = 2i + 2j + k \ .$$

a) (1 point) Compute $\vec{v} \cdot \vec{w}$	
b) (1 point) Compute $\lvert \vec{v} \rvert$	
c) (1 point) Compute $\lvert \vec{w} \rvert$	
d) (1 point) Compute $\vec{v} \times \vec{w}$	
e) (1 point) Compute $\vec{w} \times \vec{v}$	
f) (1 point) Compute $\lvert \vec{v} \times \vec{w} \rvert$	

The following problem is independent of a), b) and can be solved by ignoring quaternions:

There are two formulas involving an angle for the dot and the cross product which together assure that if both $\vec{v} \cdot \vec{w} = 0$ and $\vec{v} \times \vec{w} = \vec{0}$ then either $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$. What are they?

g) (2 points) $\vec{v} \cdot \vec{w} =$

h) (2 points) $|\vec{v} \times \vec{w}| =$

Solution:
a) 19,
b) $5\sqrt{2}$
c) 3
d) $\langle -6, 7, -2 \rangle$
e) $\langle 6, 7, 2 \rangle$
f) $\sqrt{89}$
g) $|v||w|\cos(\theta)$
h) $|v||w|\sin(\theta$

Problem 5) (10 points)

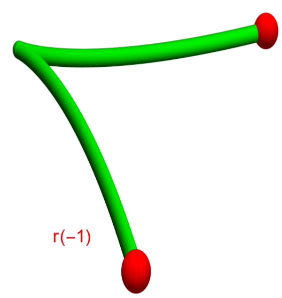
Oliver learned from his grandmother how to knit. He even made once a pullover for himself. The knitting process got him interested in topology, knots and links.

Assume that the first needle is given by $\vec{v} = \langle 1, 1, 2 \rangle$ attached at $P = (-1, 0, 0)$ and the second needle is $\vec{w} = \langle -1, 2, 1 \rangle$ attached at $Q = (1, 0, 0)$. Find the distance between the lines defined by the needles.

Solution:
Use the distance formula $|\vec{PQ} \cdot (\vec{v} \times \vec{w})|||\vec{v} \times \vec{w}|$. This is $|\langle 2, 0, 0 \rangle \cdot \langle -3, -3, 3 \rangle|/|\langle -3, -3, 3 \rangle| = 2/\sqrt{3}$.

Problem 6) (10 points)

a) (4 points) Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 12t^3/3, 1\rangle$ from $t = -1$ to $t = 1$. Hint: The integral is solvable. It is helpful to factor and split the integral.



b) (1 point) Find $\vec{r}\,'(1)$.

c) (1 point) Find $\vec{r}\,''(1)$.

d) (4 points) What is the curvature of the curve at time $t = 1$? Reminder: The formula for the curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} .$$

Solution:
a) $r\,'(t) = \langle 6t, 12t^2, 0\rangle$ and $|r\,'(t)| = 6|t|\sqrt{1+4t^2}$ Note the absolute value because the velocity is always nonnegative. The result is

$$L = 2\int_0^1 6t\sqrt{1+4t^2}\, dt = 5\sqrt{5} - 1 ,$$

where we have used integration by parts using $u = 1 + 4t^2, du = 8t$.
b) $r\,'(1) = \langle 6, 12, 0\rangle$ has length $6\sqrt{5}$.
c) We also have $r\,''(1) = \langle 6, 24, 0\rangle$. We have $r'(1) \times r''(1) = \langle 0, 0, 72\rangle$.
d) the curature is computed by taking the cross product of the vectors in b) and c) (note that its easier to compute the cross product of concrete vectors rather than doing it in general for $r(t)$).

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{|\langle 0, 0, 72\rangle|}{6^3\sqrt{5}^3} = 1/(15\sqrt{5}) .$$

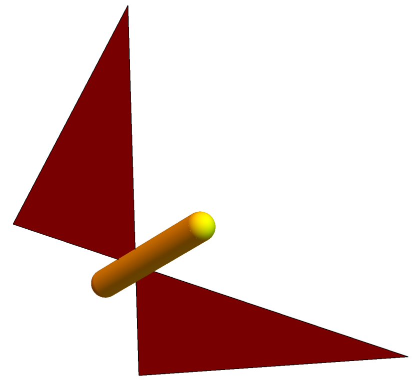
Problem 7) (10 points)

A **butterfly wing** is modeled by a simple stroke $ABCD$, where

$$A = (2, 1, 0), \quad B = (-1, 1, 0) \ ,$$

$$C = (0, 1, 2) \text{ and } D = (0, 1, -1) \ .$$

As you see in the picture, this produces two triangles touching at $O = (0, 0, 0)$. Find the wing area which is the sum of the areas of the two congruent triangles.



Solution:
The two triangles have the same area. The area of a triangle is half of the area of a parallel epiped. So, the area of the wing is actually the area of the parallel epiped spanned by OA and OB. This is the length of the cross product which is 3. $2|\vec{OA} \times \vec{OB}|/2 = 2|\langle 0, 0, 3\rangle|/2 = 3$.

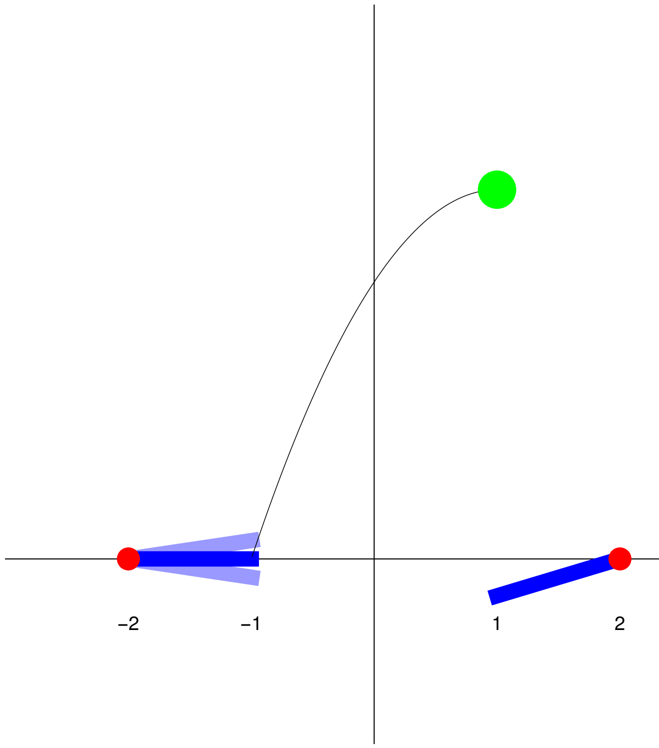
Problem 8) (10 points)

A **pinball machine** is tilted in such a way that a ball in the xy-plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}\,'(0) = \langle 1/2, 5 \rangle$.

a) (4 points) Find the velocity $\vec{r}\,'(t)$.

b) (4 points) What trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ does the ball follow?

c) (2 points) As the ball hits the line $y = 0$, is it reachable by the player? In other words, does it hit $y = 0$ within the interval $x \in [1, 2]$?



Solution:
a) We know the acceleration $\vec{r}\,''(t) = \langle 0, -2 \rangle$. Integrating once gives the velocity

$$\vec{r}\,'(t) = \langle 0, -2t \rangle + \langle 1/2, 5 \rangle \ .$$

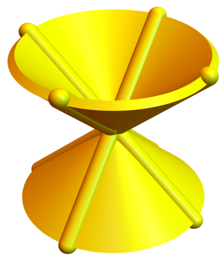
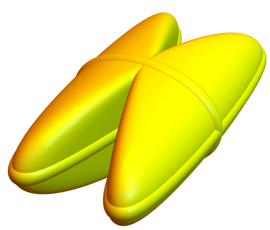
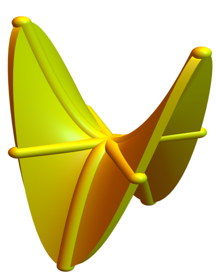
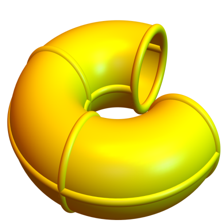
Integrating again gives

$$\vec{r}(t)\langle x(t), y(t) \rangle = \langle -1 + t/2, 5t - t^2 \rangle \ .$$

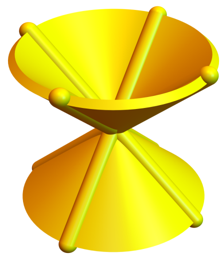
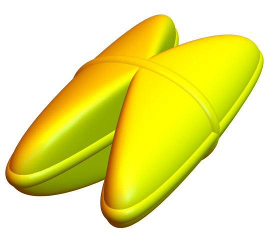
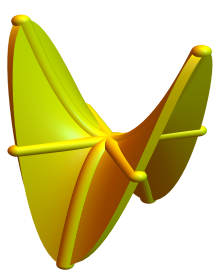
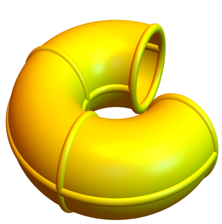
b) We have $y(t) = 0$ for $t = 0$ and $t = 5$. At this time, the ball is at $(3/2, 0)$. This is right in the middle of the right flipper. The player will hit the ball.

Problem 9) (10 points)

An MIT start-up has designed some **pasta**, which takes its shape when putting in water. This is 4D printing technology. We diversify our math-candy company and work on some pasta, parametrized by surfaces. We ask you to complete the parametrization (2 points each).

a)		The **connelloni** $\boxed{x^2+y^2=z^2}$ is parametrized by $\vec{r}(\theta,z)=\langle \dots\dots\dots\dots\dots, \dots\dots\dots\dots\dots, z\rangle$
b)		The **eightellini** $\boxed{x^2(1-x^2)=y^2}$ is parametrized by $\vec{r}(\theta,z)=\langle \cos(\theta), \sin(2\theta), \dots\dots\dots\rangle$
c)		The **lemnisagna** $\boxed{4x^2(1-z^2-x^2)=y^2(1-z^2)}$ is parametrized by $\vec{r}(\theta,\phi)=\langle \cos(\theta)\sin(\phi), \sin(2\theta)\sin(\phi), \dots\dots\dots\rangle$
d)		The **saddelloni** $\boxed{x^2-y^2=z}$ is parametrized by $\vec{r}(x,y)=\langle \dots\dots\dots, \dots\dots\dots, x^2-y^2\rangle$
e)		The **hörnli** (a Swiss pasta type) $\boxed{(\sqrt{x^2+y^2}-2)^2+z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=\langle (2+\cos(\phi))\cos(\theta), \dots\dots\dots\dots\dots, \dots\dots\dots\rangle$

Solution:

a)		The surface $\boxed{x^2 + y^2 = z^2}$ is parametrized by $\vec{r}(\theta, z) = \langle z\cos(\theta), z\sin(\theta), z\rangle$
b)		The surface $\boxed{x^2 + y^2 = 1}$ is parametrized by $\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z\rangle$
c)		The surface $\boxed{2(x-1)^2 + (y-5)^2 + 4z^2 = 1}$ is parametrized by $\vec{r}(\theta, \phi) =$ $\langle [\sin(\theta)\cos(\theta) + 1]/\sqrt{2}, 5 + \sin(\theta)\sin(\phi), \cos(\phi)/2\rangle$
d)		The surface $\boxed{x^2 - y^2 = z}$ is parametrized by $\vec{r}(x, y) = \langle x, y, x^2 - y^2\rangle$
e)		The surface $\boxed{(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1}$ is parametrized by $\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi)\rangle$