Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The directional derivative of $f(x, y)$ in the direction $\langle 1, 0 \rangle$ is $f_x(x, y)$.

Solution:
Indeed, $\nabla f \cdot \langle 1, 0 \rangle = f_x$.

2) | T | F | The surface area of the unit sphere $x^2 + y^2 + z^2 = 1$ is 4π.

Solution:
Yes, we have seen that it is the area of cylinder enclosing it. If you forgot, it can be quickly done by integrating $\sin(\phi)$ over the rectangle $[0, 2\pi] \times [0, \pi]$.

3) | T | F | The point $(0, 0)$ is a critical point of $f(x, y) = x^5 y^4$.

Solution:
Yes, $(0, 0)$ is a critical point as the gradient is zero there. We have $D = 0$ but the point still qualifies as a critical point.

4) | T | F | The function $f(x, y) = x^2 - y^2$ has a global minimum under the constraint $y = 0$.

Solution:
Indeed, we could either use Lagrange or then plug in $y = 0$ and get $f(x, y) = x^2$ on the constraint curve.

5) | T | F | The gradient ∇f of the function $f(x, y)$ with graph $z = 3x + y$ is $\langle 3, 1, -1 \rangle$.

Solution:
The gradient of a function of two variables is a vector with two components. It can not be a vector with three components.

6) T F If $(0,0)$ is a critical point for f and the second directional derivative $D_{\vec{v}}D_{\vec{v}}f(0,0)$ is positive for all unit vectors $\vec{v}$, then $(0,0)$ is a local minimum.

Solution:
Yes we have then a minimum when cutting in every direction. In a homework we have analyzed what happens with the second directional derivative if we are at a saddle point.

7) T F If $(0,0)$ is a local maximum for f, then $f_{xy}(0,0) = 0$.

Solution:
The sign of f_{xy} has no influence. Just take $x^2 + y^2 \pm xy$ which always has a minimum at $(0,0)$.

8) T F For $\vec{u} = \langle 1,1\rangle/\sqrt{2}$, we have $D_{\vec{u}}f = f_{xy}$.

Solution:
The directional derivative only invokes the first derivatives of f.

9) T F The chain rule assures that $\frac{d}{dt}H(x(t),y(t)) = H_x(x(t),y(t))x'(t) + H_y(x(t),y(t))y'(t)$.

Solution:
Yes, this is the chain rule written out in detail. [P.S. This is important in Hamiltonian dynamics where H is called the Hamiltonian. A Hamiltonian system is $x' = H_y, y' = -H_x$. In that case the Hamiltonian, the energy is conserved.]

10) T F The function $f = g^2$ under the constraint $g(x,y) = x^2 + y^2 = 1$ has never a finite set of minima.

Solution:
The constraint consists entirely of points which are minima as the function is constant on the constraint.

11) T **F** The function $u(x, y) = x^2 - y^2$ solves the partial differential equation $u_x^2 - u_y^2 = 0$.

Solution:
Just differentiate.

12) **T** F For every point (x, y) (not necessarily a critical point), there exists a direction $\vec{v}$ for which $D_{\vec{v}} f(0, 0) = 0$.

Solution:
This follows from the intermediate value theorem.

13) T **F** The identity $f_{xxx} = f_{yyy}$ holds for all smooth functions $f(x, y)$.

Solution:
This is not Clairaut at all. Clairaut deas with mixed derivatives.

14) **T** F The integral $\int_0^1 \int_0^{x^2} 1 \, dydx + \int_0^1 \int_0^{\sqrt{y}} 1 dxdy = 1$.

Solution:
Make a picture. You see that this is just the area of the unit square. Pretty cool

15) T **F** The directional derivative satisfies $D_v f = (f_{xx} f_{yy} - f_{xy}^2) = \nabla f(x, y) \cdot \vec{v}$.

Solution:
Total nonsense. The discriminant does not involve the vector $\vec{v}$ at all.

16) T **F** Fubini's theorem assures that $\int_0^1 \int_0^x f(x, y) \, dydx = \int_0^1 \int_0^y f(x, y) \, dxdy$.

Solution:
Wrong switch of order of integrtion.

17) | T | F |

When computing the surface area of the Gabriel trumpet given by $r = 1/z, z \geq 1$, we got the integral $\int_0^{2\pi} \int_1^{\infty} \frac{1}{z}\sqrt{1+z^{-4}}\, dz d\theta$.

Solution:
Indeed, that is exactly what we did.

18) | T | F |

In class, we were able to compute the integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \pi$.

Solution:
This was the part with Wilhelm Tell (the swiss bear). But the result was $\sqrt{\pi}$

19) | T | F |

The gradient vector to $z = x^2 + y^2$ at $(1, 1, 4)$ is $\langle 2x, 2y \rangle$.

Solution:
It is a 3 vector.

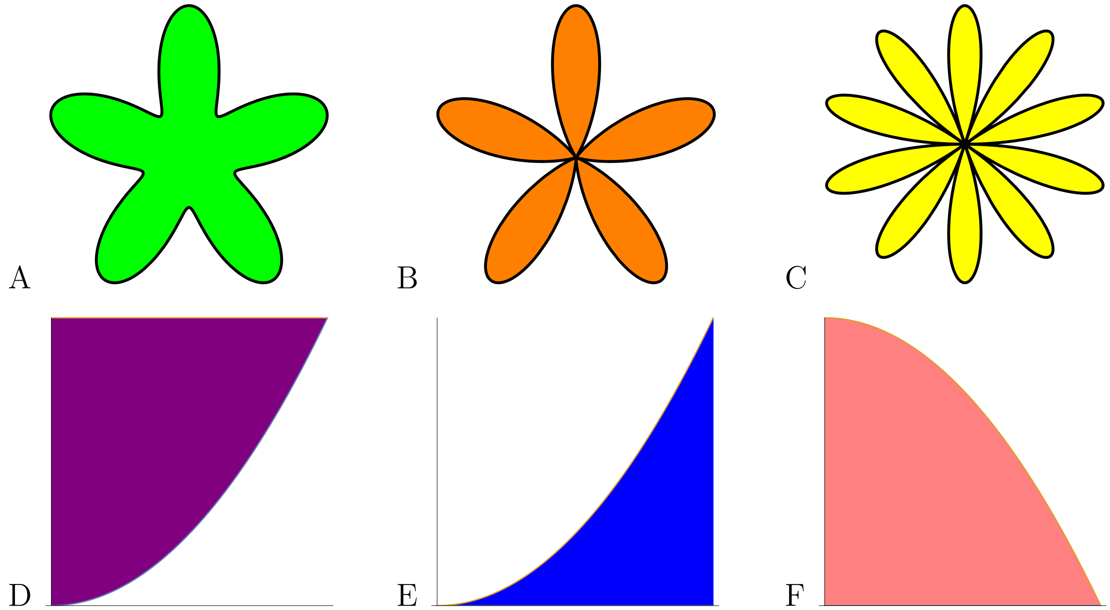
20) | T | F |

The integral $\iint_{x^2+y^2 \leq 1} |f(x, y)|\, dxdy$ computes the surface area of the surface $z = f(x, y), x^2 + y^2 \leq 1$.

Solution:
The surface area is $\sqrt{1 + f_x^2 + f_y^2}$. In general the area and the integral given are different. Take $f = 2$ for example, then the integral under consideration is 2π while the surface area is π.

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the following regions with their area computation.



Enter A-F	Area Integral
	$\int_0^{2\pi} \int_0^{1+\sin(5\theta)} r \; dr d\theta$
	$\int_0^1 \int_0^{1-x^2} 1 \; dydx$
	$\int_0^{2\pi} \int_0^{2+\sin(5\theta)} r \; dr d\theta$
	$\int_0^1 \int_{\sqrt{x}}^1 1 \; dxdy$
	$\int_0^{2\pi} \int_0^{\lvert\sin(5\theta)\rvert} r \; dr d\theta$
	$\int_0^1 \int_{x^2}^1 1 \; dydx$

b) (4 points) In the Book "In Pursuit of the Unknown", the English mathematician **Ian Stewart** covers 17 equations. Some of the are partial differential equations. Which of the following 4 equations appears in the sticky list seen in the picture? There is just one.

Fill in a)-d)	Name
	Transport
	Burgers
	Heat
	Wave

Solution:
a) BFAECD.
b) Wave

Problem 3) (10 points) (No justifications are needed.)

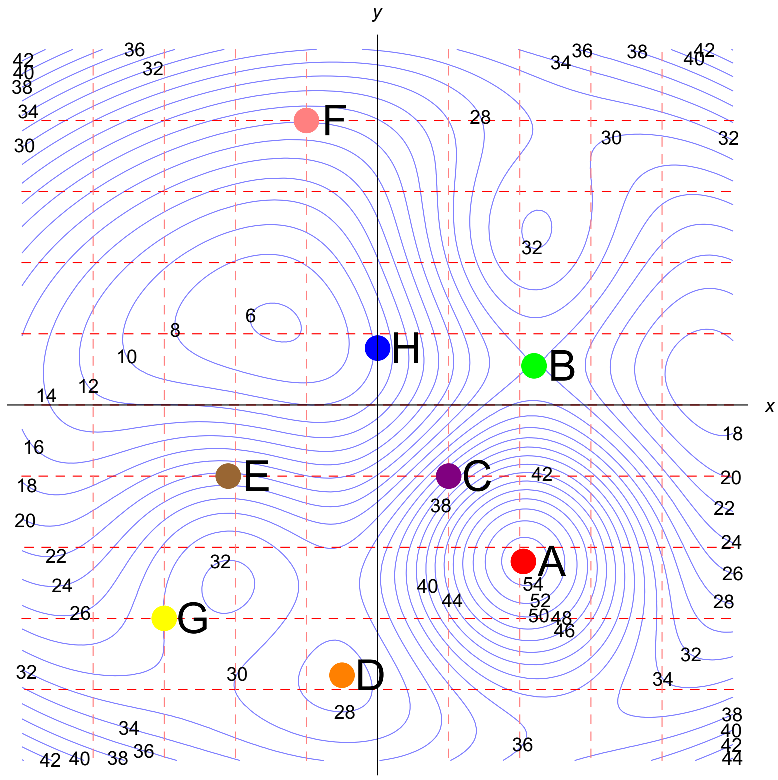
X-alps challenge is a cool alpine race which took place 2 weeks ago. The athletes had to cross the alps several times either by foot or paraglider from Insbruck to Monaco. For the 5th time, the best was the Swiss competitor **Chrigel Maurer**. He covered 2272 kilometers in 11 days. Participants have first to answer a theoretical question:

a) (2 points) Assume $g(x, y, z) = x+2y+z$ is the amount of **"thermal uplift"** at location (x, y, z) and you are at $(0, 0, 0)$ you want to go into direction, in which the uplift increase is largest. In which direction $\vec{v}/|\vec{v}|$ do you go?

A) $\vec{v} = \langle 1, 2, 1\rangle$	B) $\vec{v} = \langle -1, -2, -1\rangle$	C) $\vec{v} = \langle 2, -1, 0\rangle$	Check A)-C):	

b) (8 points) Now lets look at the terrain. In each part, pick the correct point in $A - K$. There is a possible match so that each letter appears exactly once.

	Choose one A-H
A point where $f_x = 0$ and $f_y > 0$	
A point where $f_y = 0$ and $f_x > 0$ and $f_{yy} = 0$	
A point where $\|\nabla f\|$ is maximal and $f_x > 0$	
A point where $f_x = 0$ and $f_y < 0$	
A local minimum	
A global maximum, the Matterhorn	
A saddle point	
A local minimum under the constraint $x = 0$	



Solution:
a) A
b) FGCEDABH

Problem 4) (10 points)

New England houses often feature "two slope roofs". If the length of each roof part is 1 and the angle of the upper roof is x and the angle of the lower y, then the attic room gained under the roof is

$$f(x, y) = \cos(x) + \cos(y) .$$

Assume the architect has the constraint to build it so that

$$g(x, y) = \sin(x) + \sin(y) = 1 .$$

Find the optimal x, y using the Lagrange method. We are only interested in solutions where x, y are acute angles so that there is exactly one solution. Find it.

Solution:
$x = y = Pi/6$.

Problem 5) (10 points)

We want to design an **US mailbox** for which the cost functional

$$f(x, y) = 4xy - (3\pi + 4)y^2 - 4x$$

is extremal. [The $4xy$ the surface area of the box part counting positive because it stores mail, the cylinder material and leg material parts counts negative.]

While it turns out that we can not find a minimum or maximum for f, we want to use the second derivative test to find and classify all parameters (x, y) which are critical points of f.

Solution:
There is only one critical point $(3\pi + 4)/2, 1)$. It is a saddle point as $D = -16$.

Problem 6) (10 points)

Oliver got a **lemon tree** this summer. It is seen on the picture to the right. One of the leaves is parametrized by

$$\vec{r}(x,y) = \langle x, y, y + x^2 \rangle$$

over the triangle $G : 0 \leq y \leq 1, y \leq x \leq 1$.

a) (5 points) Verify that the surface area simplifies to

$$\int_0^1 \int_y^1 \sqrt{2 + 4x^2}\; dxdy \; .$$

b) (5 points) Solve this integral! And make sure to use some of the sour power to slice that lemon.

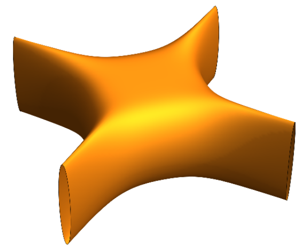
Solution:
a) Compute $r_x = \langle 1, 0, 2x \rangle$ and $r_y = \langle 0, 1, 1 \rangle$ and the cross product $\langle 2x, 1, 1 \rangle$ and its length. Then integrate.
b) Change the order of integration to get

$$\int_0^1 \int_0^x \sqrt{2 + 4x^2}\; dydx = \int_0^1 x\sqrt{2 + 4x^2}\; dx$$

which can be computed by substitution $u = 2 + 4x^2, du = 8x$ giving the result $\boxed{(1/12)[6^{3/2} - 2^{3/2}]}$.

Problem 7) (10 points)

One of the creative problems in the Mathematica project will be to create a **pasta** of your own.



a) (5 points) Find the tangent plane to the perfect *X*-**macaroni**

$$f(x,y,z) = x^2y^4 + y^2x^4 + 4z^2 = 6$$

at the point $(1,1,1)$.

b) (5 points) To taste the X-pasta, we cut it at $z = 0$ to get the curve

$$g(x,y) = x^2y^4 + y^2x^4 = 6 \ .$$

Find the tangent line at $(1,\sqrt{2})$ to this curve.

Solution:
a) Compute the gradient $\langle a,b,c\rangle = \langle 6,6,8\rangle$ then plug in $(1,1,1)$ into $ax+by+cz = d$. The answer is $\boxed{6x+6y+8z=20}$.
b) Similarly.Compute the gradient $\langle a,b\rangle = \langle 16, 10\sqrt{2}\rangle$ to get $\boxed{16x+10\sqrt{2}y=36}$.

Problem 8) (10 points)

While writing this exam, Oliver drank from an **orange soda can** of radius 1 and height 6. If the can is empty or full, the center of mass is in the middle of the can. After having tasted the juice and knowing the juice to be 4 times heaver than the can, the center of mass has gone down: there must exist a minimal value by Rolle's theorem. We simplify the problem by assuming the can is rectangular (which just changes constants). Lets find the center function $f(h)$ which depends on the **juice level height** $0 \leq h \leq 6$.

$$f(h) = \frac{\int_0^1 \int_0^6 y dy dx + 4 \int_0^1 \int_0^h y dy dx}{\int_0^1 \int_0^6 1 dy dx + 4 \int_0^1 \int_0^h 1 dy dx}$$

a) (4 points) Compute each of the four double integral in this expression.
b) (3 points) What is $f(h)$? You don't need to simplify except if you want to make it easier for c).
c) (3 points) Evaluate the $f(h)$ values for $h = 0, h = 3$ and $h = 6$ to check that $f(0) = f(6)$ is indeed in the middle of the can and that $f(3)$ is smaller.

P.S. ignore the product placement for 4 brands on this picture. You also don't have to find the minimum h. It turns out to be the golden ratio $(\sqrt{5} - 1)/2$ times the height. Golden juice + Golden ratio = Golden summer!

Solution:
a) The integrals are $9, 2h^2, 6, 4h$.
b) The magic formula is $(9 + 2h^2)/(6 + 4h)$.
c) Just plug in the values for h. We get $3, 2, 3$.

Problem 9) (10 points)

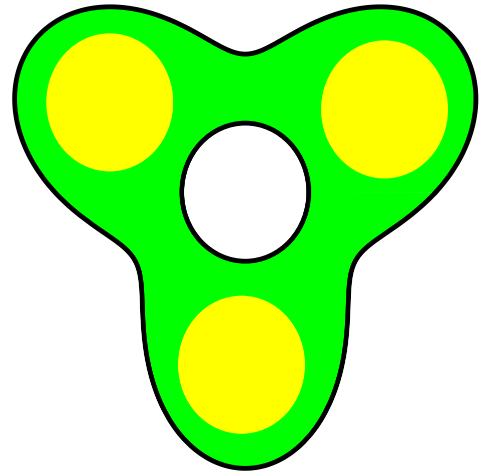
a) (5 points) Find the volume below the graph of the function $f(x, y) = x^4 + y^4$ and above the square G given by $-1 \leq x \leq 1, -1 \leq y \leq 1$. In other words, find

$$\int\int_G f(x, y)\ dxdy\ .$$

b) (5 points) 2017 was the year of the **fidget spinner**! What is the moment of inertia

$$\int\int_G x^2 + y^2\ dxdy$$

of the **fidget spinner region** G given in polar coordinates as $1 \leq r \leq 3 + \sin(3\theta)$?



Solution:
a) Integrate directly (no polar coordinates). We get 8/5.
b) Here we use polar coordinates :

$$\int_0^{2\pi} \int_0^{3+\sin(\theta)} r^3\ drd\theta\ .$$

This is $(1/4) \int_0^{2\pi} (3 + \sin(3\theta))^4 - 1 d\theta$. By the Binomial formula we have $(3 + \sin(3\theta))^4 = 3^4 + 4 \cdot 3^3 \sin(3\theta) + 6 \cdot 3^2 \sin^2(3\theta) + 4 \cdot 3 \sin^3(3\theta) + \sin^4(3\theta)$. Integrating $\sin(3\theta)$ or $\sin^3(\theta)$ gives zero. For the $\sin^4(\theta)$ part write it as $\sin^2(3\theta)(1 - \cos^2(3\theta)$. The result is $859\pi/4$.