

Name:

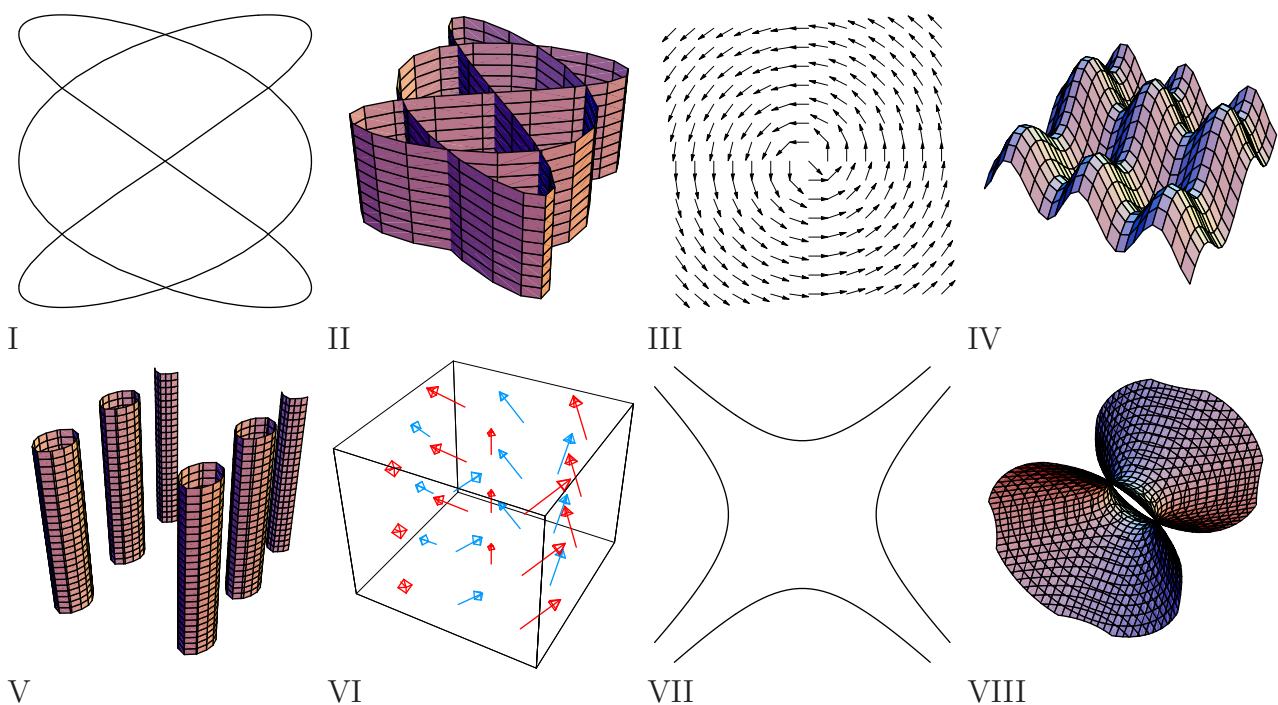
- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
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11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

- | | | |
|-----|---|---|
| 1) | ☐ T ☐ F | If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$ then $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$. |
| 2) | ☐ T ☐ F | $\int_0^5 \int_0^\pi r \, d\theta \, dr$ is half the area of a disc radius 5 in the plane. |
| 3) | ☐ T ☐ F | If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = 0$ for all points (x, y) in the plane, then $\vec{F}$ is conservative. |
| 4) | ☐ T ☐ F | If the acceleration of a parameterized curve $\vec{r}(t) = (x(t), y(t), z(t))$ is zero then the curve $\vec{r}(t)$ is a line. |
| 5) | ☐ T ☐ F | A circle of radius $1/2$ has a smaller curvature than a circle of radius 1. |
| 6) | ☐ T ☐ F | The curve $\vec{r}(t) = (-\sin(t), \cos(t))$ for $t \in [0, \pi]$ is half a circle. |
| 7) | ☐ T ☐ F | The function $u(t, x) = \sin(x + t)$ is a solution of the partial differential equation $u_{tx} + u = 0$ |
| 8) | ☐ T ☐ F | The length of a curve $\vec{r}(t)$ in space parameterized on $a \leq t \leq b$ is the value of the integral $\int_a^b \vec{T}'(t) \, dt$, where $\vec{T}(t)$ is the unit tangent vector. |
| 9) | ☐ T ☐ F | Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then the gradient of g at (x_0, y_0) is parallel to the gradient of f at (x_0, y_0) . |
| 10) | ☐ T ☐ F | At a point which is not a critical point, the directional derivative $D_{\vec{v}}f(x_0, y_0, z_0)$ can take both the negative and the positive sign. |
| 11) | ☐ T ☐ F | If a nonzero vector field $\vec{F}(x, y)$ is a gradient field, we always can find a curve C for which the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive. |
| 12) | ☐ T ☐ F | If C is a closed level curve of a function $f(x, y)$ and $\vec{F} = (f_x, f_y)$ is the gradient field of f , then $\int_C \vec{F} \cdot d\vec{r} = 0$. |
| 13) | ☐ T ☐ F | The divergence of a gradient vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is always zero. |
| 14) | ☐ T ☐ F | The line integral of the vector field $\vec{F}(x, y, z) = [x^2, y^2, z^2]^T$ along a line segment from $(0, 0, 0)$ to $(1, 1, 1)$ is 1. |
| 15) | ☐ T ☐ F | If $\vec{F}(x, y) = (x^2 - y, x)$ and $C : \vec{r}(t) = [\sqrt{\cos(t)}, \sqrt{\sin(t)}]^T$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C \vec{F} \cdot d\vec{s}$ is twice the area of R . |
| 16) | ☐ T ☐ F | The flux of the vector field $\vec{F}(x, y, z) = [0, y, 0]^T$ through the boundary S of a solid sphere E is equal to the volume the sphere. |
| 17) | ☐ T ☐ F | The quadratic surface $-x^2 + y^2 + z^2 = 5$ is a one-sheeted hyperboloid. |
| 18) | ☐ T ☐ F | If $\vec{F}$ is a vector field in space and S is the boundary of a solid sphere then the flux of $\text{curl}(\vec{F})$ through S is 0. |
| 19) | ☐ T ☐ F | If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) and S is a torus surface, then the flux of $\vec{F}$ through S is zero. |
| 20) | ☐ T ☐ F | In spherical coordinates, the equation $\rho \cos(\phi) = \rho \cos(\theta) \sin(\phi)$ defines a plane. |

Problem 2) (10 points)



Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$x^2 - y^2 + z^2 = 1$
	$\vec{r}(t) = [\cos(3t), \sin(2t)]^T$
	$z = f(x, y) = \cos(3x) + \sin(2y)$
	$\vec{F}(x, y) = [-y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2}]^T$
	$\cos(3x) + \sin(2y) = 1$
	$\vec{F}(x, y, z) = [-y, x, 1]^T$
	$\vec{r}(u, v) = [\cos(3u), \sin(2u), v]^T$
	$\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 1\}$

Furthermore, fill in the peoples names, Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$	
$f_{xy}(x, y) = f_{yx}(x, y)$	
$\int_C \vec{F} \cdot d\vec{r} = \int \int_R \text{curl}(\vec{F}) \, dx dy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int \int_S F \cdot d\vec{S} = \int \int \int_E \text{div}(F) \, dV$	
$\int_a^b \int_c^d f(x, y) \, dx dy = \int_c^d \int_a^b f(x, y) \, dy dx$	

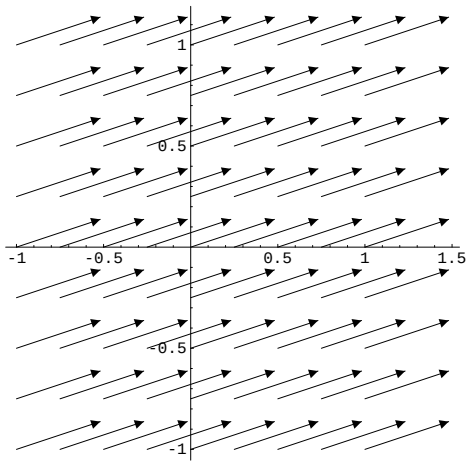
Problem 3) (10 points)

In this problem, vector fields $\vec{F}$ are written as $\vec{F} = [P, Q]^T$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x, y) = 0$ vanishes for **all** (x, y) . Similarly, we say $\text{div}(F)$ if $\text{div}(F)(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all x, y .

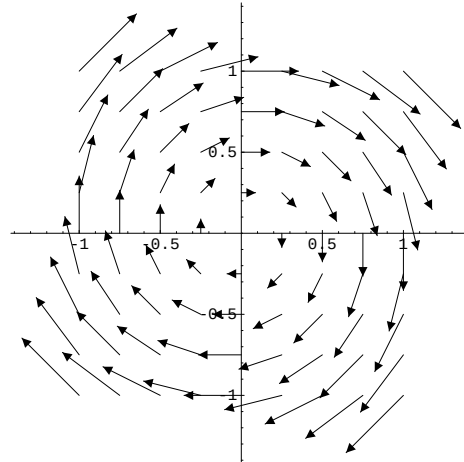
Check the box which match the formulas of the vector fields with the corresponding picture I, II, III or IV and mark also the places, indicating the vanishing of $\text{curl}(F)$.

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{div}(F) = 0$
$\vec{F}(x, y) = [1, x]^T$						
$\vec{F}(x, y) = [3y, -3x]^T$						
$\vec{F}(x, y) = [7, 2]^T$						
$\vec{F}(x, y) = [x, y]^T$						

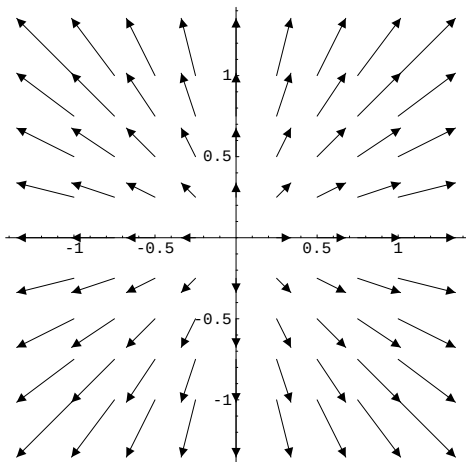
I



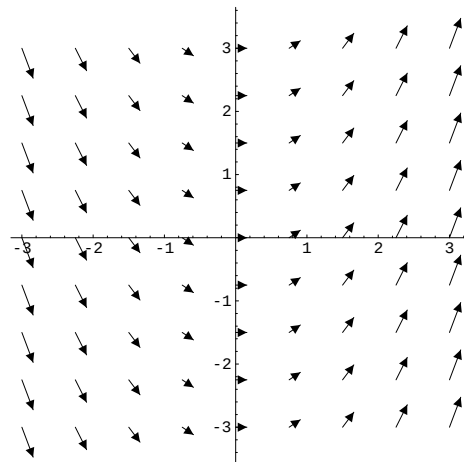
II



III



IV



Problem 4) (10 points)

- a) (5 points) What is the area of the triangle A, B, P , where $A = (1, 1, 1)$, $B = (1, 2, 3)$ and $P = (3, 2, 4)$?
- b) (5 points) Find the distance between the point P and the line L passing through the points A with B .

Problem 5) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x, y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2}.$$

There is a lake in that area as you can see in the photo.

- a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.
- b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \cdot \vec{dr}$.



Photo of the lake in the Swiss alps near the Simplon mountain pass.

Problem 6) (10 points)

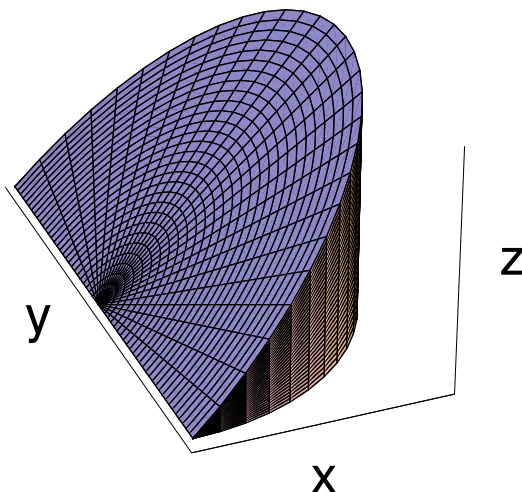
- a) (4 points) Find the linearization $L(x, y, z)$ of $f(x, y, z) = 2 + z - \sin(-x - 3y)$ at the point $P = (0, \pi, 2)$.

b) (4 points) Find the equation of the tangent plane at that point $P = (0, \pi, 2)$.

c) (2 points) Estimate $f(0.001, \pi, 2.02)$ using the linearization.

Problem 7) (10 points)

Find the volume of the wedge shaped solid that lies above the xy -plane and below the plane $z = x$ and within the solid cylinder $x^2 + y^2 \leq 9$.

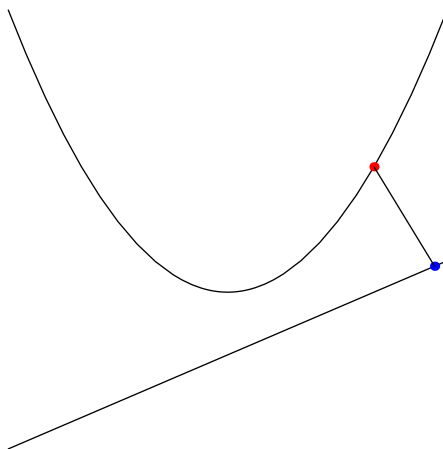


Problem 8) (10 points)

The distance from a point (x, y) to the line $y = x$ in the plane is given by $f(x, y) = (y - x)/\sqrt{2}$. Use the Lagrange method to find the point (x, y) on the parabola

$$g(x, y) = x^2 - y = -2$$

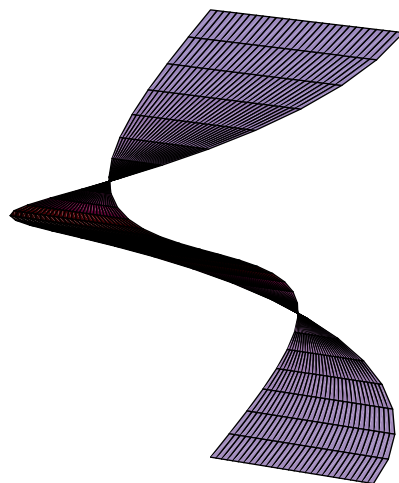
which is closest to the line.



Problem 9) (10 points)

a) (5 points) A ribbon of a girl is modeled as a surface S which is parameterized by $\vec{r}(t, s) = [s \cos(t), \sin(t), t]^T$, where $t \in [0, 2\pi]$ and $s \in [0, 1]$. Find the surface area of this ribbon S .

b) (5 points) Part of the boundary of the ribbon is obtained when fixing $s = 1$. It is a curve in space. Find the arc length of this curve $\vec{r}(t)$, parametrized from $t = 0$ to 2π .

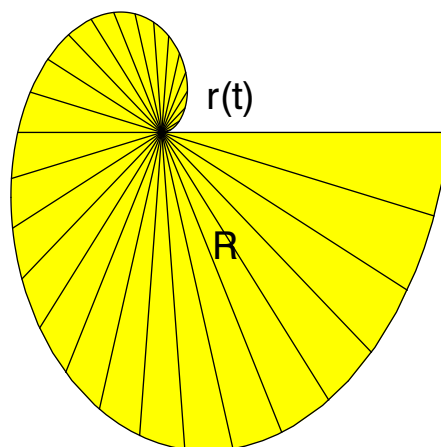


Painting: "Young Girl with Blue Ribbon" by the French painter Jean-Baptiste Greuze (1725-1805)

Problem 10) (10 points)

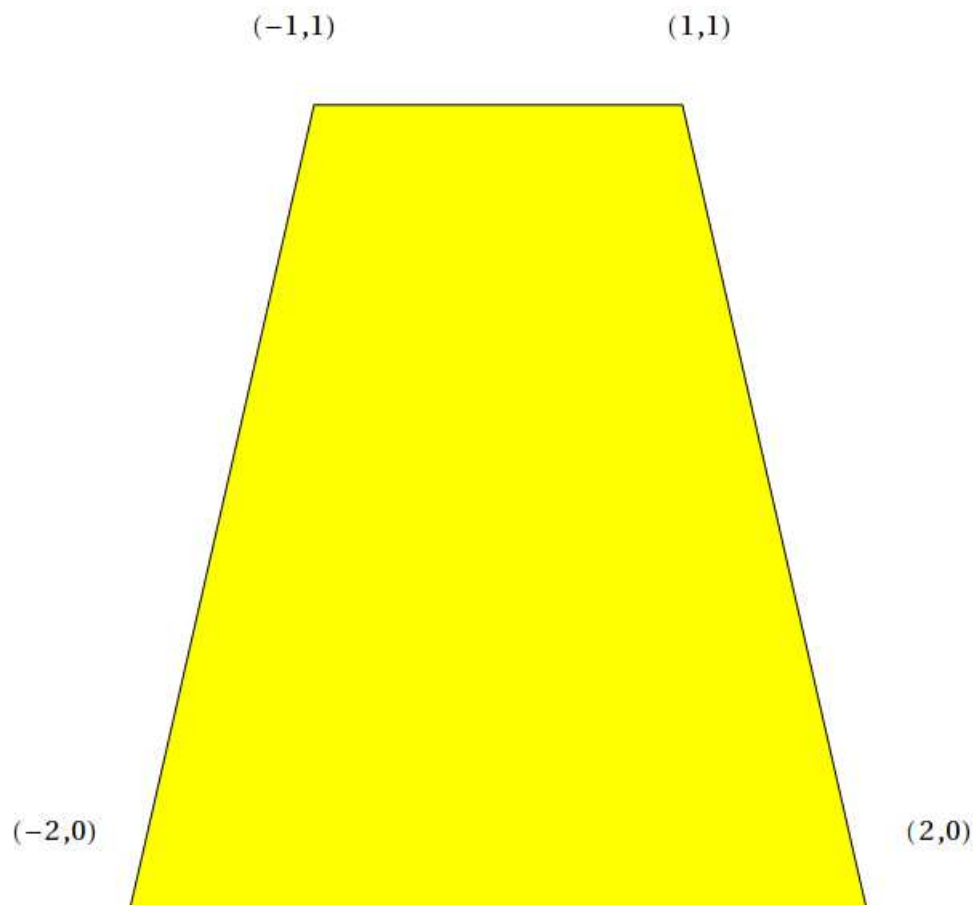
A region R in the xy -plane is given in polar coordinates by $0 \leq r(\theta) \leq \theta$ for $\theta \in [0, 2\pi]$. You see the region in the picture to the right. Its boundary is called the **Archimedes spiral**. It can be found on the tomb of Jacob Bernoulli. Evaluate the double integral

$$\iint_R \frac{e^{-x^2-y^2}}{(2\pi - \sqrt{x^2 + y^2})} dx dy .$$



Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x, y) = [3y, 8x]^T$ along the boundary of the trapezoid with vertices $(-2, 0), (2, 0), (1, 1), (-1, 1)$.



Problem 12) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = [-z + x^{(x)}, 5 + y^{(y)}, y + z^{(z)}]^T$.
 Let C be the curve given by the parameterization $\vec{r}(t) = [\cos(t), 0, \sin(t)]^T$, for $0 \leq t \leq 2\pi$.
 Compute the line integral of $\vec{F}$ along C .

Hint. You might want to consider a surface contained in the xz -plane which is enclosed by the curve.

Problem 13) (10 points)

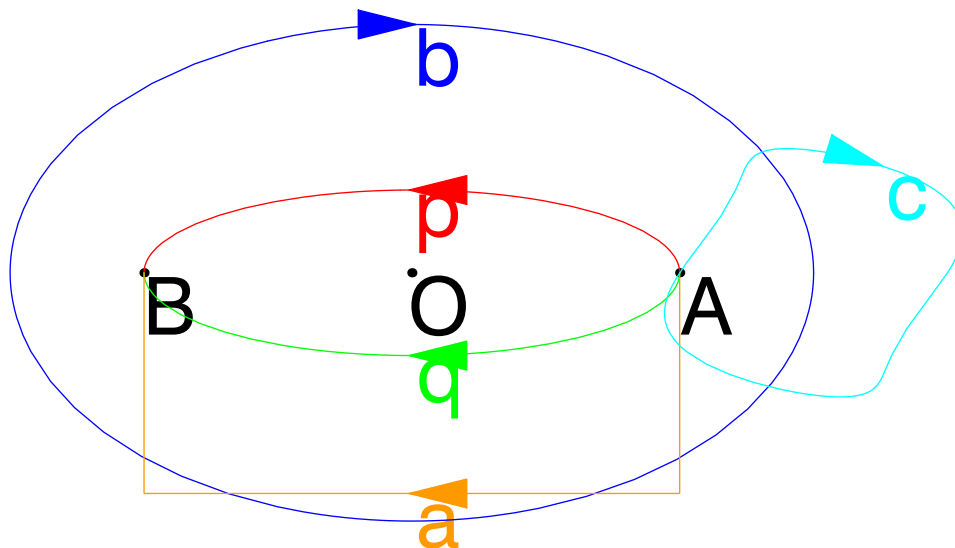
What is the flux of the vector field

$$\vec{F}(x, y, z) = [3x + \cos(z^2 \sin(z)), x, \sin(y^3 + \cos(\sin(xy^3)))]^T$$

through the boundary S of the solid cylinder $E = \{x^2 + y^2 \leq 1, 0 \leq z \leq 10\}$. The surface of the cylinder is oriented so that the normal vector points outwards.

Problem 14) (10 points)

Suppose $\vec{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. Suppose the line integral of $\vec{F}$ along the path p from A to B is 5 and the line integral of $\vec{F}$ along the path q from A to B is -4 . Find the line integral of $\vec{F}$ along the following three paths:



- a) (3 points) The path a from A to B going clockwise below the origin.
- b) (4 points) The closed path b encircling the origin in a clockwise direction.
- c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Problem 15) (10 points)

Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy -plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int \int_S \vec{F} \cdot d\vec{S}$ of the vectorfield

$$\vec{F} = \left(-4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2}\right)$$

through S using the divergence theorem.