Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-2 do not require any justifications. Problem 3 only 1-2 words. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

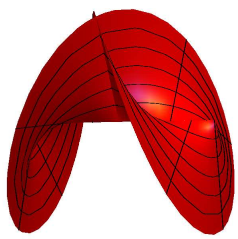
1)	T	F	The equation $G_{xx}G_{yy} - G_{xy}^2 = 1$ is an example of a partial differential equation
2)	T	F	The curvature of a curve is $\kappa(t) = \|\vec{r}\,'(t) \times \vec{r}\,''(t)\|/\|\vec{r}\,'(t)\|$.
3)	T	F	If S is the boundary of a solid E and $\int\int\int_E 1 \; dV = 1$, then $\int\int_S \|r_u \times r_v\| \; dudv = 1$.
4)	T	F	The cross product of two nonzero perpendicular vectors has length 0.
5)	T	F	The function $f(x,y,z) = \|\text{curl}(\vec{F}(x,y,z))\|$ is maximal, where the divergence of $\vec{F}$ is zero.
6)	T	F	The distance between two circles of radius 1 in space is equal to the distance between the centers plus 2.
7)	T	F	For an ellipse contained in the xz-plane, the binormal vector $\vec{B}$ is always perpendicular to that plane.
8)	T	F	The surface $x^2 - y^2 - z^2 = 1$ is a one sheeted hyperboloid.
9)	T	F	Let $G = \text{curl}(\vec{F})$. A flow line of the vector field $\vec{G}$ is always a circle.
10)	T	F	The cross product of two unit vectors $\vec{v}$ and $\vec{w}$ has length ≤ 1.
11)	T	F	The unit tangent vector $\vec{T}'(t)$ always has length 1.
12)	T	F	The discriminant $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ is always smaller than the length of $\nabla f(x,y)$ at the point.
13)	T	F	The curl of an incompressible vector field is zero.
14)	T	F	$\vec{r}(u,v) = [u^3, 2, u^3 + v]^T$ parametrizes a plane.
15)	T	F	The integral $\int\int\int_E \|\text{curl}(\vec{F}(x,y,z))\| \; dxdydz$ is zero for all solids E.
16)	T	F	If two flow lines of a vector field $\vec{F}$ intersect at a nonzero angle, then the vector field $\vec{F}$ is zero at that point.
17)	T	F	For any vector field $\vec{F}$ we have $\text{div}(\text{grad}(\text{div}(\vec{F}))) = 0$ at every point.
18)	T	F	The triple scalar product is always positive.
19)	T	F	If $\vec{v} = \nabla f / \|\nabla f\|$ at a non-critical point, then the directional derivative of f in the direction of $\vec{v}$ is positive.
20)	T	F	The vector projection $\vec{P}_{\vec{v}}(\vec{w})$ is a vector which has length smaller or equal than the length of the vector $\vec{w}$.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the following surfaces. There is an exact match.

Surface	Enter 1-4
$\vec{r}(u,v) = [\cos(u), \cos(v), \sin(u) + v]^T$	
$\vec{r}(t,s) = [4\sin(t), s\sin(2t), s\cos(2t)]^T$	
$\vec{r}(t,s) = [s\cos(t), s\sin(t+s), s]^T$	
$x^4 + y^4 + z^4 - 6x^2 = 0.3$	

1



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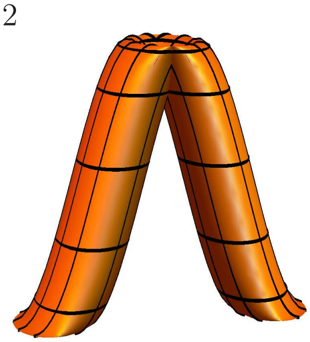

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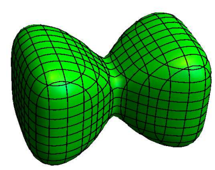

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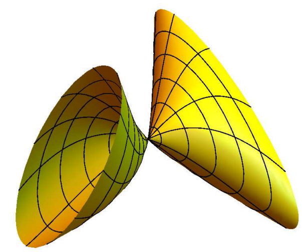


b) (3 points) Match the integral types with the names. There is an exact match.

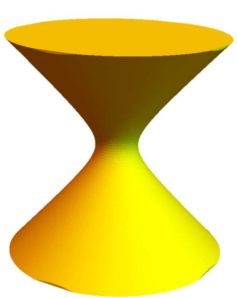
Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert\, dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$	
$\int_a^b \int_c^d \lvert\vec{r}_u \times \vec{r}_v\rvert\, dudv$	

A	line integral
B	flux integral
C	arc length
D	surface area

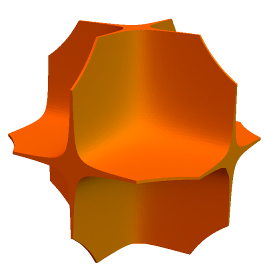
c) (3 points) Match the solids. There is an exact match.

Solid	Enter a-d
$\lvert xyz\rvert \leq 1$	
$x - y^2 + z^2 \leq 1$	
$x^2 + y^2 - z^2 \leq 1$	
$1 \leq x^2 + y^2 \leq 2$	

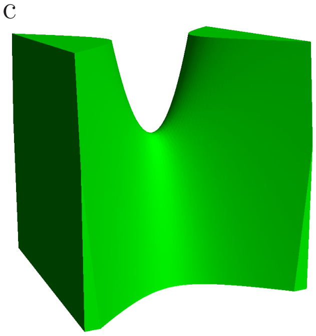
a



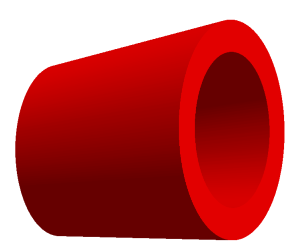
b



c



d



Problem 3) (10 points) Only one word justifications necessary

Physics Nobel prize winner **Richard Feynman** (we have seen him in a scene of the movie "Infinity" in class), has written a book called "6 easy pieces". Inspired by the title, we write 6 problems:

a) (1.666 points) What is the flux of $\vec{F} = [x, 2y, 3z]^T$ through the unit sphere $x^2 + y^2 + z^2 = 1$, oriented outwards?

Answer: because of:

b) (1.666 points) What is double integral $\int\int_G \text{curl}(\vec{F})\, dxdy$, where G is the unit disk $\{x^2+y^2 \leq 1\}$ and $\vec{F} = [-3y, 2x]^T$?

Answer: because of:

c) (1.666 points) What is the flux of the curl of the vector field $\vec{F} = [x^2, y^3, z]^T$ through the surface $x^2 + y^4 + z^6 = 1$ oriented outwards?

Answer: because of:

d) (1.666 points) What is the line integral of $\vec{F} = [x^3, y^5, z^6]^T$ along the curve $\vec{r}(t) = [\cos(t), \sin(t), \cos(3t)]^T$ parametrized from $t = 0$ to $t = 2\pi$?

Answer: because of:

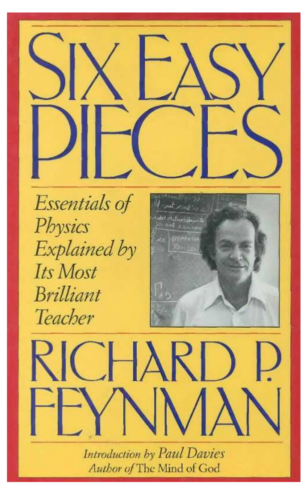
e) (1.666 points) What is the flux of the curl of the vector field $\vec{F} = \nabla f$ through the upper ellipsoid $x^2/4 + y^2/9 + z^2/25 = 1, z \geq 0$?

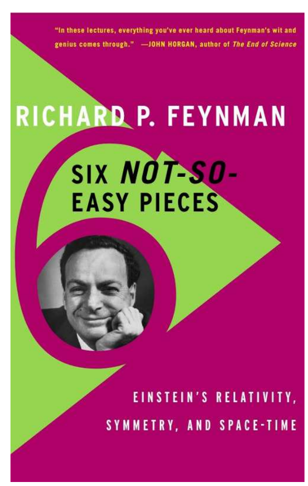
Answer: because of:

f) (1.666 points) What is the line integral of the vector field $\vec{F}(x, y) = [1, 0]^T$ along the curve $\vec{r}(t) = [\cos(t), \sin(t)]^T$ from $t = 0$ to $t = \pi$?

Answer: because of:

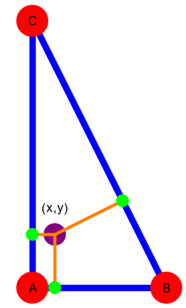
Why use fractional points? First of all, because it is more fun to grade, second because fundamental particles like quarks also can have fractional charge. We will use the formula $P = \min(10, 2C)$ for the points P earned in this problem, where C is the number of correct answers.
By the way, there is an other book coauthored by Feynman called "Six not so easy pieces". That might also apply to the problems above.





Problem 4) (10 points)

Given the triangle $A = (0,0,0), B = (1,0,0), C = (0,2,0)$, we aim to find the point $P = (x,y,0)$ for which the sum of the squares of the distances to the triangle edges is minimal.



a) (3 points) Give a formula for the distance of $P = (x,y,0)$ to the line through the points B and C.

b) (7 points) The sum of the distance squares multiplied by 5 turns out to be

$$f(x,y) = 5x^2 + 5y^2 + (2x + y - 2)^2 .$$

Find the critical points of $f(x,y)$ and classify them.

Problem 5) (10 points)

We all adored "monkey" in the movie "Hangover II". On the picture you see him sitting on his **Monkey saddle** $z = f(x,y) = x^3 - 3xy^2$ which is called "Phil".

a) (4 points) Find the tangent plane to $g(x,y,z) = z - f(x,y) = 0$ at $(2,1,2)$.

b) (4 points) Estimate $2.001^3 - 3 \cdot 2.001 \cdot 0.99^2$.

c) (2 point) Find the directional derivative $D_{\vec{v}} f$ at $(2,1)$ if $\vec{v} = [0,1]^T$.

Problem 6) (10 points)

This is the last chance to show that you are not a zombie!

a) (5 points) Evaluate the double integral

$$\int_0^3 \int_{x^2}^9 \frac{x}{e^{y^2}} \, dy dx \; .$$

b) (5 points) Integrate the curl of $\vec{F}(x,y) = [-yx^2, xy^2]^T$ over the region $4 \le x^2 + y^2 \le 9$.

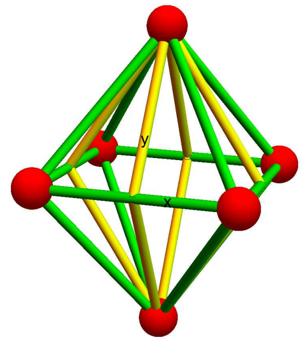
Problem 7) (10 points)

A **diamond** has the shape of an octahedron. Assume the side length of the equator is x and the side height y. Extremize the surface area

$$f(x,y) = 4xy$$

of the diamond under the constraint that the total length of the supporting height pieces is constant

$$g(x,y) = 4x + 8y = 16 \ .$$



Problem 8) (10 points)

Yesterday on August 6th, 2014, the **Rosetta space craft** arrived at the 2.5 miles wide Churyumov-Garasimenko comet. This coming November, a lander **Philae** will set down on the comet by harpooning itself to the surface and becoming the first space craft landing on a comet. Assume the deceleration is $\vec{r}\,''(t) = [-t, -t^2, 0.1]^T$ which is a combination of thruster force and microgravity and that $\vec{r}\,(0) = [3,4,3]^T$ and $\vec{r}\,'(0) = [0,0,1]^T$. Find the path $\vec{r}(t)$ of the space craft.

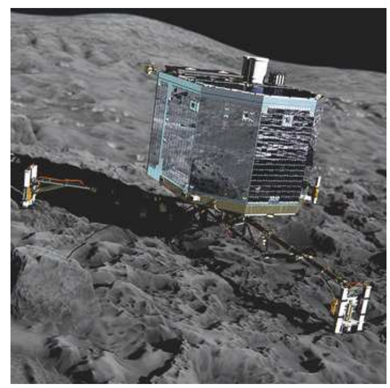


Image Credit: European Space Agency: http://www.esa.int

Problem 9) (10 points)

Warka water towers are a recent invention. They can pull gallons of fresh water out of thin air. Assume the vector field of the surrounding air is $\vec{G} = \text{curl}(\vec{F})$, where

$$\vec{F}(x,y,z) = [x + \sin(z), -2y + z^5 \sin(5z), z]^T \ .$$

Find the flux of $\vec{G}$ through the surface of revolution given as $S : r(z) = 2 + \sin(z)$ and $0 \leq z \leq \pi$. Note that the surface does not include the bottom. You compute the amount of fluid gathered by the tower in a day.

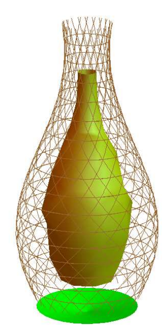


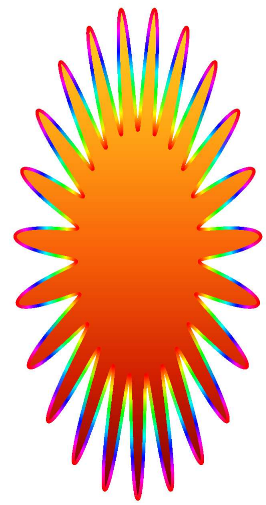
Image credit for first picture: http://www.smithsonianmag.com.

Problem 10) (10 points)

Use an integral theorem to find the area of the region enclosed by the curve

$$\vec{r}(t) = [(3+\sin(23t))\cos(t), 2(3+\sin(23t))\sin(t)]^T$$

from $t = 0$ to $t = 2\pi$. As seen several times in class, you can use that integrating an odd 2π periodic function from 0 to 2π is zero.



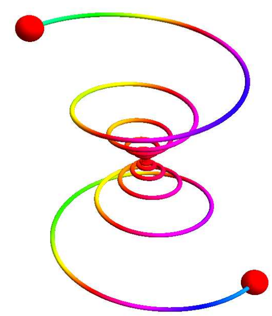
By the way, if you stare for an hour at the picture with your nose 3 inches from the paper, you get hallucinations as the curve is an example of a **hypnagogic stimulus curve**. *It only works with "23". It can also be used as a* **Rorschach test**: *what do you see when you look at the inkblot?*

Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x, y, z) = [2x, 3y^2, 3z^3]^T$ along the **coil** curve

$$\vec{r}(t) = [t\cos(3\pi\log(|te|)), t\sin(3\pi\log(|te|)), t]^T$$

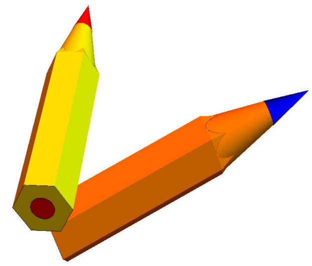
from $t = -1$ to $t = 1$, where as usual, log is the natural logarithm and $e = \exp(1)$ is the Euler's mathematical constant.

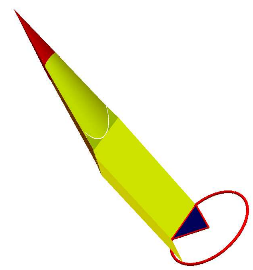


By the way, this curve has finite length but dances around the z-axes infinitely often. Great idea for a roller coaster!

Problem 12) (10 points)

When Oliver learned calculus he was given the problem to find the volume of the **pencil** E, a hexagonal cylinder of radius 1 above the xy-plane cut by a sharpener below the cone $z = 10 - x^2 - y^2$. Still having nightmares about this, he needs therapy to get rid of this "pencil sharpener phobia": we consider one sixth of the pen where the base is the polar region $0 \leq \theta \leq 2\pi/6$ and $r(\theta) \leq \sqrt{3}/(\sqrt{3}\cos(\theta) + \sin(\theta))$. The pen's back is $z = 0$ and the sharpened part is $z = 10 - r^2$. Write down the triple integral for the volume of E. You get full credit for writing down the correct integral.





P.S. If you have spare time, try to solve the integral but it is probably the "hardest math problem in the world. If somebody of you can solve it, I will make sure that 'none of you will ever have to open a math book again'." (As you know, this line is stolen from the movie "Rushmore".)

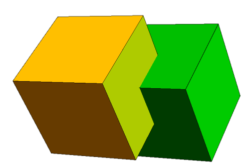
Problem 13) (10 points)

A computer can determine the volume of a solid enclosed by a triangulated surface by computing the flux of the vector field $\vec{F} = [0, 0, z]^T$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = [0, 0, z]^T$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2\} \ \cup$$

$$\{1 \leq x \leq 3, 1 \leq y \leq 3, 1 \leq z \leq 3\} \ ,$$

where $\cup$ is the union and the surface is oriented outwards.



Ceci n'est pas une pipe

Ceci c'est une vache