

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

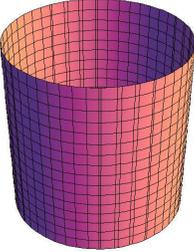
Problem 1) (20 points) No justifications are necessary

- 1) T F The dot product between $[1, 1, 2]^T \cdot [2, 3, 4]^T$ is 13.
- 2) T F The distance between a line $\vec{r}(t) = \vec{Q} + t\vec{v}$ with unit vector \vec{v} and a point P is given by $|\vec{PQ} \times \vec{v}|$.
- 3) T F The TNB frame consists of the vectors $\vec{r}_u, \vec{r}_v, \vec{r}_u \times \vec{r}_v$.
- 4) T F The surface $x^2 + y^2 - z^2 = -1$ is a one sheeted hyperboloid.
- 5) T F The equation $u_{tt} + uu_x = u_{xx}$ is called the Burgers equation.
- 6) T F The fundamental theorem of line integrals assures that the line integral of any vector field along a closed loop is zero.
- 7) T F If $\vec{r}(t)$ is a curve in space then $\nabla\vec{r}(t)$ is a vector perpendicular to the curve.
- 8) T F If $\nabla f(3, 1) = [0, 0]^T$ and $f_{xx}(3, 1) > 0$, then $(3, 1)$ is a local minimum of f .
- 9) T F Given two points P, Q in space and two lines L, M where L goes through P and M goes through Q . The distance between P, Q is larger or equal than the distance between the two lines.
- 10) T F The equation $u_t - u_x = u_{xx}$ is called the heat equation.
- 11) T F The flux of the curl of \vec{F} through the surface S is positive, where S is the surface $x^2 + y^2 + z^2 = 1$ oriented outwards.
- 12) T F The dot product between two parallel vectors is always zero.
- 13) T F $\vec{r}(u, v) = [u, u, 0]^T$ parametrizes a surface S which is a cylinder.
- 14) T F The length of the gradient $|\nabla f|$ is always minimal at critical points.
- 15) T F The triple integral $\int \int \int_E \text{div}(\vec{F}(x, y, z)) \, dx dy dz$ over a sphere E is always zero since the flux of \vec{F} through the boundary surface is zero.
- 16) T F Assume $\vec{r}(t)$ is a path of length 1 parametrized on $[a, b]$, then $\int_a^b |\vec{r}'(t)| \, dt = 1$.
- 17) T F If $\vec{r}'(t) = [2t, 1 - 2t]^T$ and $\vec{r}(0) = [2, 3]^T$, then $\vec{r}(t) = [2 + t^2, 3 + t - t^2]^T$.
- 18) T F For any two unit vectors \vec{v}, \vec{w} we have $|\vec{v} \times \vec{w}|^2 + (\vec{v} \cdot \vec{w})^2 = 1$.
- 19) T F The directional derivative $D_v(f)$ is always perpendicular to the vector \vec{v} and to the surface $f = c$.
- 20) T F $[1, 0, 0]^T \cdot ([1, 1, 0]^T \times [0, 0, 1]^T) = 1/6$.

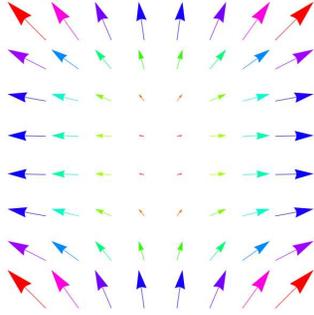
Problem 2) (10 points) No justifications are necessary.

Match the following objects.

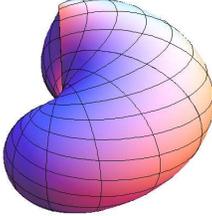
Formula	Enter 1-9
$\rho \leq (1 + \sin(\phi + \theta))$	
$x^2 + y^2 + z^2 + \sin(x^2y^2z^2) = 1.$	
$r = t + (2\pi - t) + \cos(5\theta), z = 0.$	
$\vec{F}(x, y) = [x, y^2]^T$	
$(x - 5)^2 + (y - 3)^2 + (z + 1)^2 = 1$	
$x^2 + y^2 = 3$	
$\vec{r}(t) = [\cos(t), \sin(3t), \sin(2t)]^T$	
$\vec{r}(u, v) = [(3 + \sin(u) \cos(v)) \cos(u), (3 + \sin(u) \cos(v)) \sin(u), (1 + \sin(u)) \sin(v)]^T$	
$\vec{F}(x, y, z) = [x, y^2, z^3]^T$	



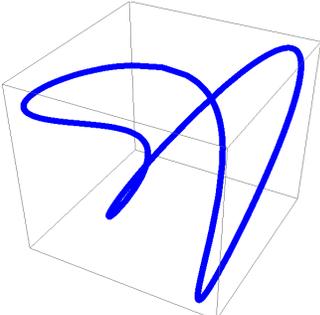
1



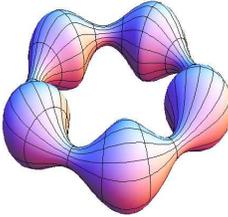
2



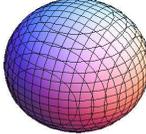
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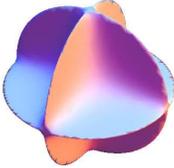
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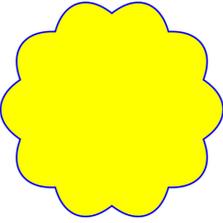
5



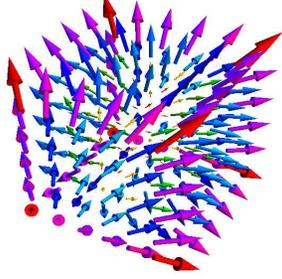
6



7



8



9

Problem 3) (10 points) No justifications are necessary

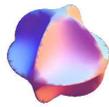
a) (3 points) Matching solids.

Formula	Enter A,B,C,D
$x^2 + y^2 + z^2 \leq 1$	
$x^{40} + y^{40} + z^{40} \leq 1$	
$x^2 y^2 z^2 + x^2 + y^2 + z^2 \leq 1$	
$x^2 + y^2 \leq 1, z^2 \leq 1$	

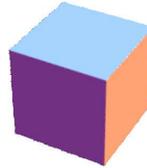
A



B



C

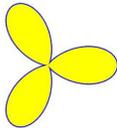


D



b) (3 points) Matching polar regions

Formula	Enter E,F,G,H
$r = \theta$	
$r = \sin(3\theta) $	
$r = \sin(\theta)^6$	
$r = 1 + \cos(3\theta)$	



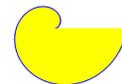
E



F



G



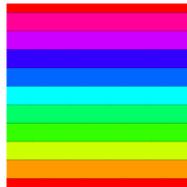
H

c) (3 points) Matching level curves

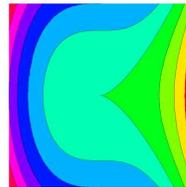
Formula	Enter I,J,K,L
$f(x, y) = 5y - 1$	
$f(x, y) = x^4 - y^4$	
$f(x, y) = 3x^2 + y^2$	
$f(x, y) = y^2 - x^3$	



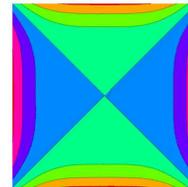
I



J



K



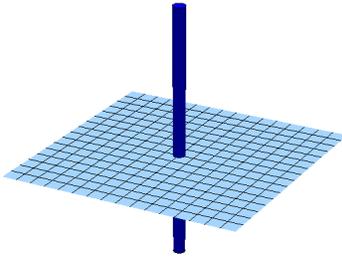
L

d) (1 point) Partial differential equation

The equation $f_{tt} = f_{xx}$ is called

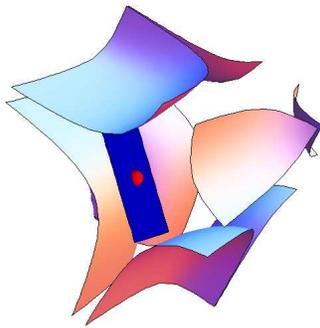
Heat equation	
Wave equation	
Transport equation	
Burgers equation	

Problem 4) (10 points)



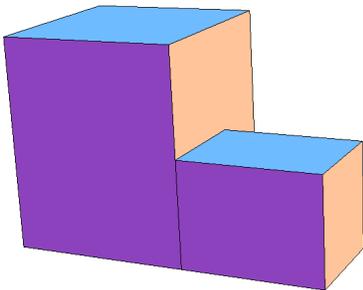
- a) (3 points) Find a formula for the distance of a point (x, y, z) to the xy -plane.
- b) (3 points) Find a formula for the distance of a point (x, y, z) to the z -axis.
- c) (4 points) Find the surface consisting of all points (x, y, z) for which the distance to the z -axis is the same than the distance to the xy plane.

Problem 5) (10 points)



- a) (5 points) Estimate $2.001^3 \cdot 0.9999^4 \cdot 0.999^2$ using linearization.
- b) (5 points) Find the tangent plane to the surface $x^3y^4z^2 = 8$ at the point $(2, 1, 1)$.

Problem 6) (10 points)



A **chicken coop** is made of two cubes of length x and y . The volume of the house is $f(x, y) = x^3 + y^3$. The surface area is $g(x, y) = 5x^2 + 3y^2$. Using Lagrange, find the coop of maximal volume if the constraint is $g = 38$.

Problem 7) (10 points)

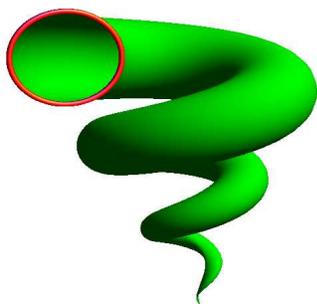


The roof of the tower of the Harvard **Lovell house** has height

$$f(x, y) = 1 - (x^2 + y^2)^7 .$$

Find the volume under the roof above the disc $x^2 + y^2 \leq 1$ in the xy -plane.

Problem 8) (10 points)



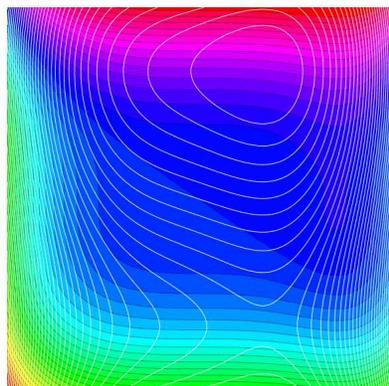
What is the flux of the curl of the field $\vec{F}(x, y, z) = [0, z^2 + x^4, x]^T$ through the shell

$$[s(2 + \cos(t)) \cos(s), s(2 + \cos(t)) \sin(s), 6s + s \sin(t)]^T ,$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq 6\pi$. The shell has the boundary curve

$$\vec{r}(t) = [6\pi(2 + \cos(t)), 0, 36\pi + 6\pi \sin(t)]^T .$$

Problem 9) (10 points)



At which point does the function

$$u(x, y) = \frac{2x^3}{3} + 2y^3 - \frac{x^6}{30} - \frac{y^5}{20}$$

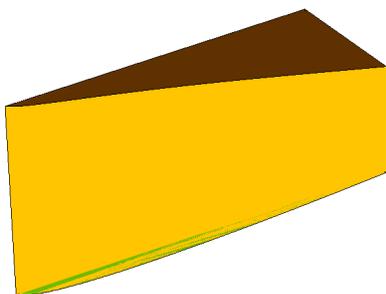
have the property that

$$f(x, y) = u_{xx}(x, y) + u_{yy}(x, y)$$

is extremal. Find and classify all the critical points of $f(x, y)$.

Remark you can ignore: this problem appears in physics. If the function u is the **electric potential**, then f is the **charge density**. You find the place where the charge density is maximal.

Problem 10) (10 points)



The following integral gives the volume of a piece of **Swiss cheese**

$$\int_0^3 \int_{\sqrt{y/3}}^1 \int_0^{e^{-x^3}} 1 \, dz \, dx \, dy .$$

Find it.

Problem 11) (10 points)



While Mars rover “**Curiosity**” was landing on Mars, a force

$$\vec{F}(x, y, z) = [\sin(x), y, -30z]^T$$

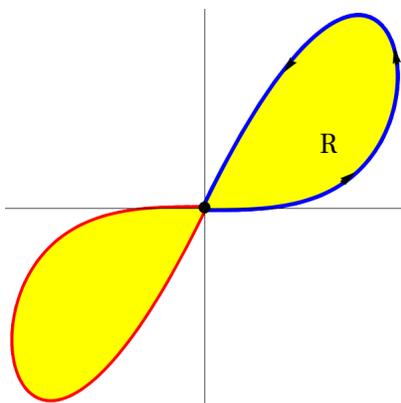
acted on the rover while it was ‘descending on the path

$$\vec{r}(t) = [1, 2t, 10 - t^2]^T .$$

Find the line integral

$$\int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt .$$

Problem 12) (10 points)

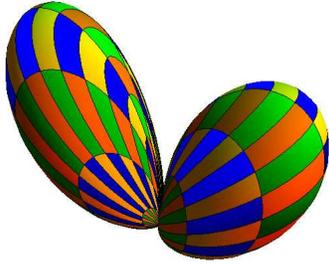


Find the area of the **propeller** shaped region enclosed by the figure 8 curve

$$\vec{r}(t) = [t - t^3, 2t^3 - 2t^5]^T ,$$

parametrized by $-1 \leq t \leq 1$. To find the total area compute the area of the region R enclosed by the right loop $0 \leq t \leq 1$ and multiply by 2.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x, y, z) = [-y^7, -x^8, -z + x^5]^T$$

through the surface given in spherical coordinates as

$$\rho \leq (\sin(\phi) \cos(\phi) \cos^2(\theta))^{1/3}$$

with $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$. The surface is oriented outwards.