Name:

- Start by printing your name in the above box.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.

- Do not detach pages from this exam packet or unstaple the packet.

- Please try to write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids are allowed.

- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) [T] [F]   The lines $\vec{r_1}(t) = [t, t, -t]^T$ and $\vec{r_2}(t) = [1+t, 1+t, 1-t]^T$ do not intersect.

2) [T] [F]   The quadratic surface $x^2 - y^2 = z^2$ is a hyperbolic paraboloid.

3) [T] [F]   If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}\,'(t) \neq 0$ everywhere, then they span a parallelepiped of volume 1.

4) [T] [F]   If $\vec{u} \cdot \vec{v} = 0$, then $\mathrm{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

5) [T] [F]   There is a vector field $\vec{F}(x, y)$ which has the property $\mathrm{curl}(\vec{F}) = -\mathrm{div}(\vec{F})$, where $\mathrm{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$ and $\mathrm{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$.

6) [T] [F]   The acceleration vector $\vec{r}\,''(t) = [x(t), y(t), z(t)]^T$ is always a unit vector if the velocity vector $\vec{r}\,'(t)$ is a unit vector.

7) [T] [F]   The grid curves $t \to \vec{r}(t, \phi)$ with fixed $0 < \phi < \pi$ for the standard parametrization of the unit sphere have curvature $1/\sin(\phi)$.

8) [T] [F]   Any smooth function $f(x, y)$ has a local maximum somewhere in the plane.

9) [T] [F]   The linearization $L(x, y)$ of constant function $f(x, y) = 3$ is $L(x, y) = 3$.

10) [T] [F]   A gradient field is incompressible: it satisfies $\mathrm{div}(F) = 0$ everywhere.

11) [T] [F]   If $f(x, y)$ has a maximum under the constraint $g(x, y) = 1$, then $\nabla f = [0, 0]^T$ at this point.

12) [T] [F]   Assume a vector field $\vec{F}(x, y, z)$ is the curl of a vector field $\vec{G}$ then the flux of the field F through the ellipsoid $x^2 + y^2 + 5z^2 \leq 1$ is zero.

13) [T] [F]   If the divergence of a field $\vec{F}$ are zero everywhere, then any line integral along a closed curve is zero.

14) [T] [F]   The gradient of the divergence of a field is always the zero field.

15) [T] [F]   The vector field $\vec{F}(x, y, z) = [x^2, y^2, z^3]^T$ is a gradient field.

16) [T] [F]   The volume of a solid can be computed as the flux of the field $[0, y, 0]^T$ through the boundary surface.

17) [T] [F]   The curvature of a line is zero.

18) [T] [F]   The distance between the unit sphere centered at $(0, 0, 0)$ and the plane $z = 5$ is equal to 4.

19) [T] [F]   The partial differential equation $u_t = u_x$ is called heat equation.

20) [T] [F]   The point $(1, -1, \sqrt{2})$ in spherical coordinates is $(\rho, \phi, \theta) = (2, \pi/4, 3\pi/2)$.
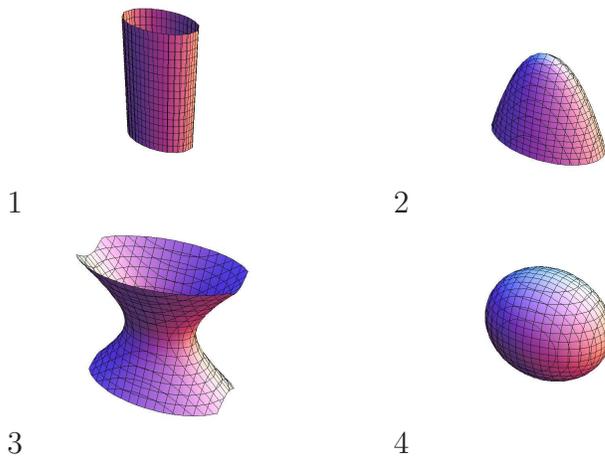
2

Problem 2) (10 points) No justifications are necessary.

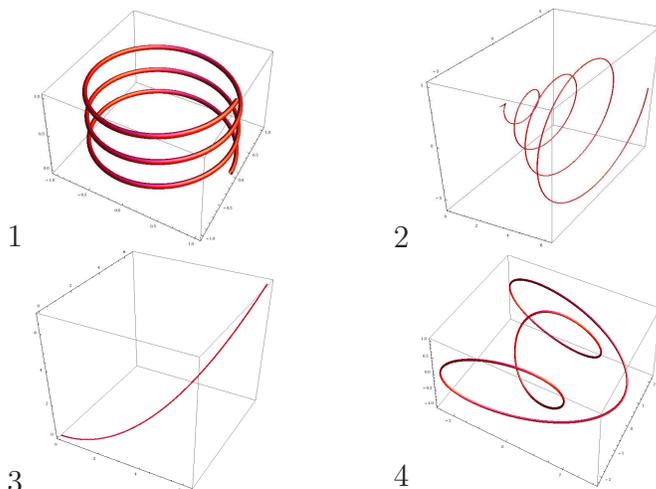a) (4 points) Match the objects with the definitions.

1

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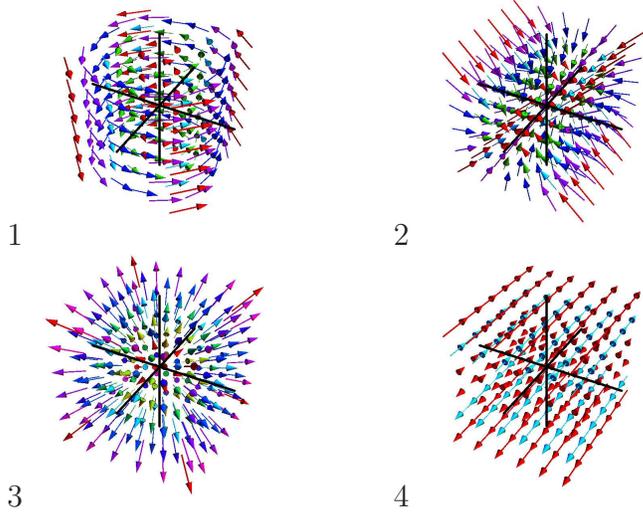
enter 1-4	vector field
	$\vec{F}(x,y,z) = [x,y,z]^T$
	$\vec{F}(x,y,z) = [-y,x,0]^T$
	$\vec{F}(x,y,z) = [0,z,0]^T$
	$\vec{F}(x,y,z) = [-x,0,-z]^T$

b) (3 points) Match the surfaces with their names: (put O if no match)

1

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4

enter 1-4	surface
	$x^2 + y^2 + 3z = 0$
	$x^2 + y^2 - 3z^2 = 1$
	$x^2 + y^2 + 3z^2 = 1$
	$x^3 + 3y^2 = 1$

c) (3 points) Match the space curves

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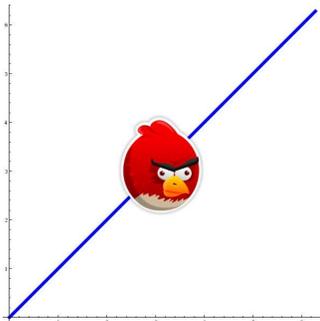| 1-4	parametrized curve
	$\vec{r}(t) = [t, t^2, t^3]^T$
	$\vec{r}(t) = [\cos(3t), \sin(3t), t]^T$
	$\vec{r}(t) = [(2 + \cos(t))\cos(3t),$ $(2 + \cos(t))\sin(t),$ $\sin(3t)]^T$
	$\vec{r}(t) = [t, t\cos(3t), t\sin(3t)]^T$

Problem 3) (10 points) No justifications are necessary

a) (5 points) We watch "angry birds" attacking on curves with acceleration $\vec{r}\,''(t)$. (The pictures show the $xz-$ planes and the birds start with a constant velocity $[1,0,0]^T$.) Match the displayed curves $\vec{r}(t)$ with the formulas for accelerations.
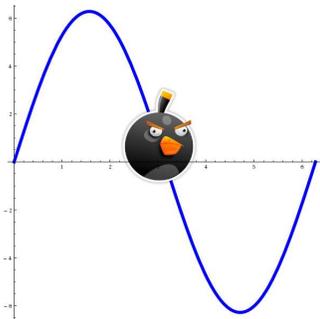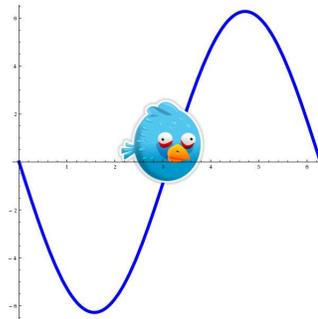
1

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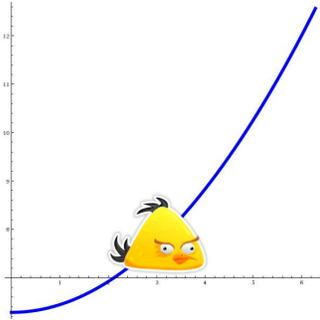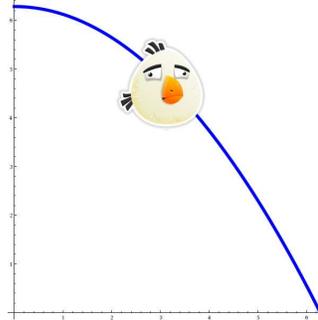
acceleration	enter curve 1-5
$\vec{r}\,''(t) = [0,0,\sin(t)]^T$	
$\vec{r}\,''(t) = [0,0,-10]^T$	
$\vec{r}\,''(t) = [0,0,10]^T$	
$\vec{r}\,''(t) = [0,0,-\sin(t)]^T$	
$\vec{r}\,''(t) = [0,0,0]^T$	

3

4

5

b) (5 points) Match the formulas: (put O if no match)

label	formula
A	$\vec{r}\,'(t)$
B	$\int_0^1 |\vec{r}\,'(t)|\ dt$
C	$\vec{r}\,'(t)/|\vec{r}\,'(t)|$
D	$\vec{T}\,'(t)/|\vec{T}\,'(t)|$
E	$|\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$

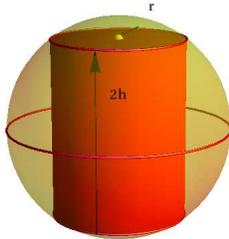| expression	enter A-E
Curvature	
Unit tangent vector	
Unit normal vector	
Velocity	
Arc length	

4

Problem 4) (10 points)
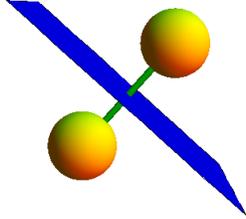
a) (5 points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y - 1)^2 + z^2 = 1$, $(x - 5)^2 + y^2 + z^2 = 1$.

b) (5 points) Find the plane perpendicular to the line L for which the distances to the spheres are the same.
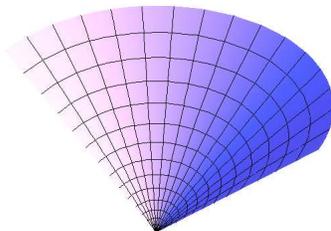
Problem 5) (10 points)

Johannes Kepler asked which cylinder or radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve his problem, use the Lagrange method and maximize the volume

$$f = 2\pi r^2 h$$

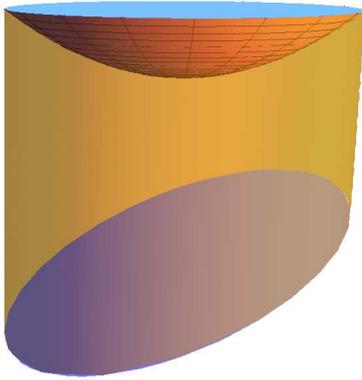under the constraint that $r^2 + h^2 = 1$.

Problem 6) (10 points)

a) (6 points) Find the surface area of the surface

$$r(u, v) = [v^2 \cos(u), v^2 \sin(u), v^2]^T, 0 \le u \le \pi, 0 \le v \le 1 \,.$$

b) (4 points) Find the arc length of the boundary curve $\vec{r}(u, 1)$ where $0 \le u \le \pi$.
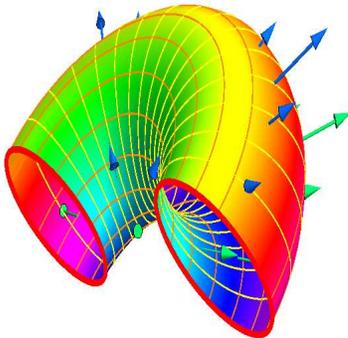
Problem 7) (10 points)

Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 + 4$.

Problem 8) (10 points)

Find the flux of the curl of the vector field

$$\vec{F}(x, y, z) = [x, y, z + \sin(\sin(y^2))]^T$$

through the torus

$$\vec{r}(s, t) = [(2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s)]^T$$

with $0 \leq t \leq \pi$ and $0 \leq s < 2\pi$.
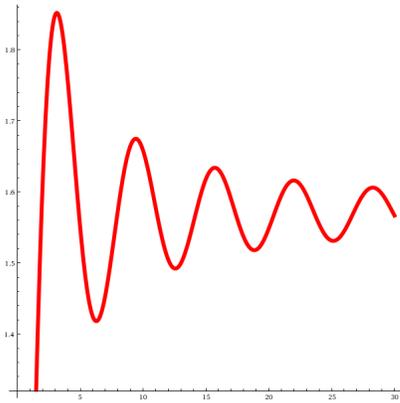
Problem 9) (10 points)

Heron's formula for the area A of a triangle of side length $x, y, 1$ satisfies $16A^2 = f(x, y)$, where

$$f(x, y) = -1 + 2x^2 - x^4 + 2y^2 + 2x^2y^2 - y^4 .$$

Classify all the critical points of f. Is there a global maximum of f and so for the area?

Remark not to worry about: The formula follows directly from Heron's formula $s = (a + b + 1)/2; A = \sqrt{s(s-a)(s-b)(s-1)}$.
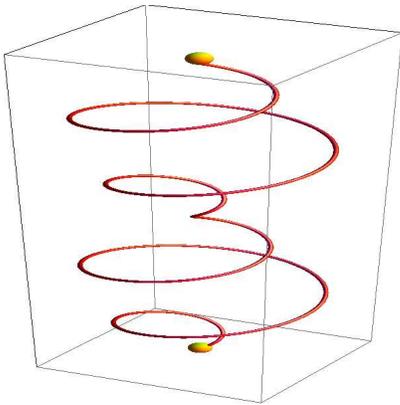
The anti derivative of the **sinc** function
$$\frac{\sin(x)}{x}$$
is called the **sine integral** $\text{Si}(x)$. It can not be expressed in terms of known functions. Still we can compute the following double integral
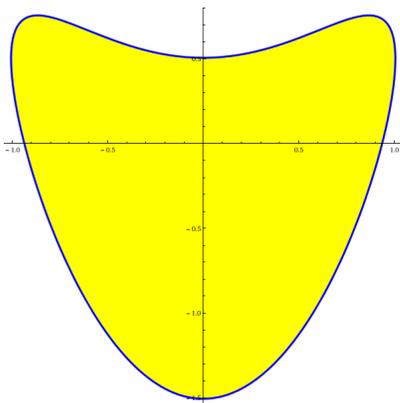$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} \, dy \, dx \ .$$

Problem 11) (10 points)

Find the line integral of the vector field
$$\vec{F}(x, y, z) = [-x^{10}, \sin(y), z^3]^T$$
along the curve $\vec{r}(t) = [\sin(t)\cos(5t), \sin(t)\sin(5t), t]^T$ where $0 \le t \le 2\pi$.
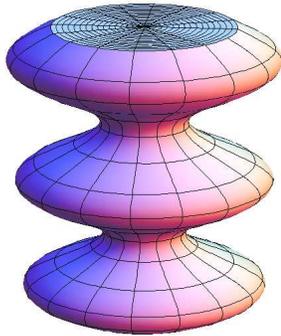
Problem 12) (10 points)

Find the area of the region enclosed by the curve
$$\vec{r}(t) = [\cos(t), \sin(t) + \cos(2t)/2]^T \ ,$$
where $0 \le t < 2\pi$.

Find the flux of the vector field

$$\vec{F}(x, y, z) = [x^3/3, y^3/3, \sin(xy^5)]^T$$

through the boundary surface of the solid bound by the surface of revolution $\vec{r}(t, z) = [(2 + \sin(z))\cos(t), (2 + \sin(z))\sin(t), z]^T$ and the planes $z = 0$, $z = 3$. The surface is oriented so that the normal vector points outwards.