

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

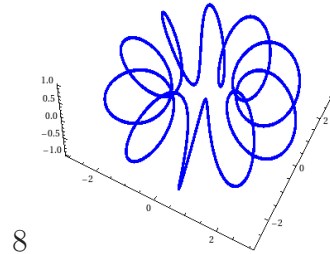
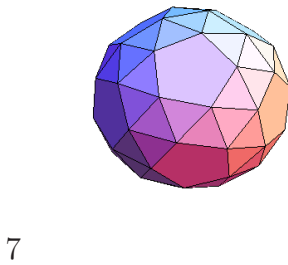
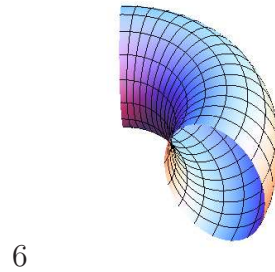
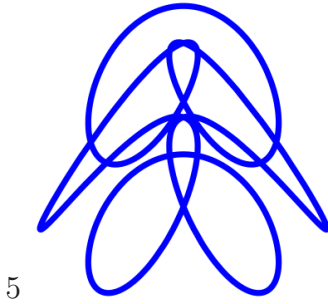
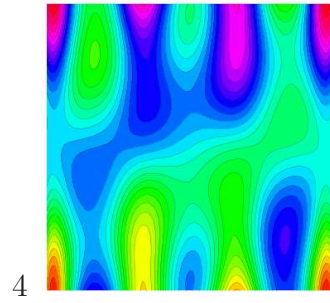
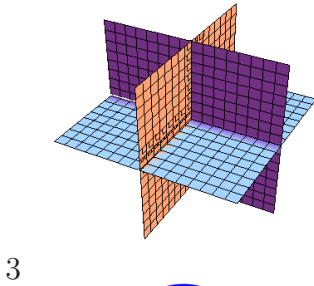
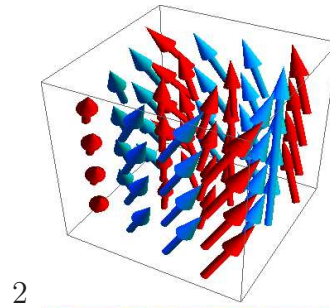
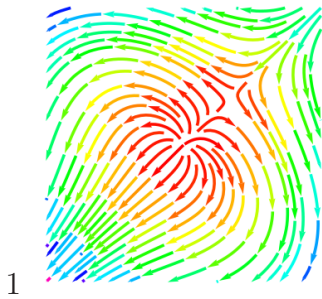
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2		10
3		10
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6		10
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8		10
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10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

- 1) ☐ T ☐ F The line $\vec{r}(t) = [t, t, t]^T$ is perpendicular to the plane $x + y + z = 10$.
- 2) ☐ T ☐ F The quadratic surface $-x^2 + y^2 + z^2 = -1$ is a one sheeted hyperboloid.
- 3) ☐ T ☐ F The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.
- 4) ☐ T ☐ F $\int_0^{\pi/2} \int_0^1 r^3 d\theta dr = \int_0^1 \int_0^1 x^2 + y^2 dxdy$.
- 5) ☐ T ☐ F If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) = 0$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all points (x, y) in the plane, then $\vec{F}$ is a constant field.
- 6) ☐ T ☐ F The acceleration vector $\vec{r}''(t) = [x(t), y(t)]^T$, the velocity vector $\vec{r}'(t)$ and $\vec{r}'(t) \times \vec{r}''(t)$ form three vectors which are mutually perpendicular.
- 7) ☐ T ☐ F The curvature of the curve $\vec{r}(t) = [\sin(2t), 0, \cos(2t)]^T$ is equal to the curvature of the curve $\vec{s}(t) = [0, \cos(3t), \sin(3t)]^T$.
- 8) ☐ T ☐ F The space curve $\vec{r}(t) = [t \sin(t), t \cos(t), t^2]^T$ for $t \in [0, 10\pi]$ is located on a cone.
- 9) ☐ T ☐ F If a smooth function $f(x, y)$ has a global maximum, then this maximum is a critical point.
- 10) ☐ T ☐ F If $L(x, y)$ is the linearization of $f(x, y)$ and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ at t_0 . Then $d/dt L(\vec{s}(t)) = d/dt f(\vec{r}(t))$ at the time $t = t_0$.
- 11) ☐ T ☐ F If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.
- 12) ☐ T ☐ F If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, we have infinitely many extrema.
- 13) ☐ T ☐ F If a point (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a critical point of the function $f(x, y)$ without constraints.
- 14) ☐ T ☐ F If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any ellipse is zero.
- 15) ☐ T ☐ F The flux of an irrotational vector field is zero through any surface S in space.
- 16) ☐ T ☐ F The divergence of a gradient field $\vec{F}(x, y) = \nabla f(x, y)$ is zero.
- 17) ☐ T ☐ F The line integral of a vector field $\vec{F}(x, y, z) = [x, y, z]^T$ along a circle in the xy - plane is zero.
- 18) ☐ T ☐ F For any solid E , the moment of inertia $\iiint_E x^2 + y^2 dxdydz$ is always larger than the volume $\iiint_E 1 dxdydz$.
- 19) ☐ T ☐ F The curvature of a circle is always larger than the acceleration.
- 20) ☐ T ☐ F The directional derivative of $\text{div}(\vec{F}(x, y))$ of the divergence of the vector field $\vec{F} = [P, Q]^T$ in the direction $\vec{v} = [1, 0]^T$ is $P_{xx} + Q_{xy}$.

Problem 2) (10 points) Match objects with definitions. No justifications necessary.

Match the objects with their definitions



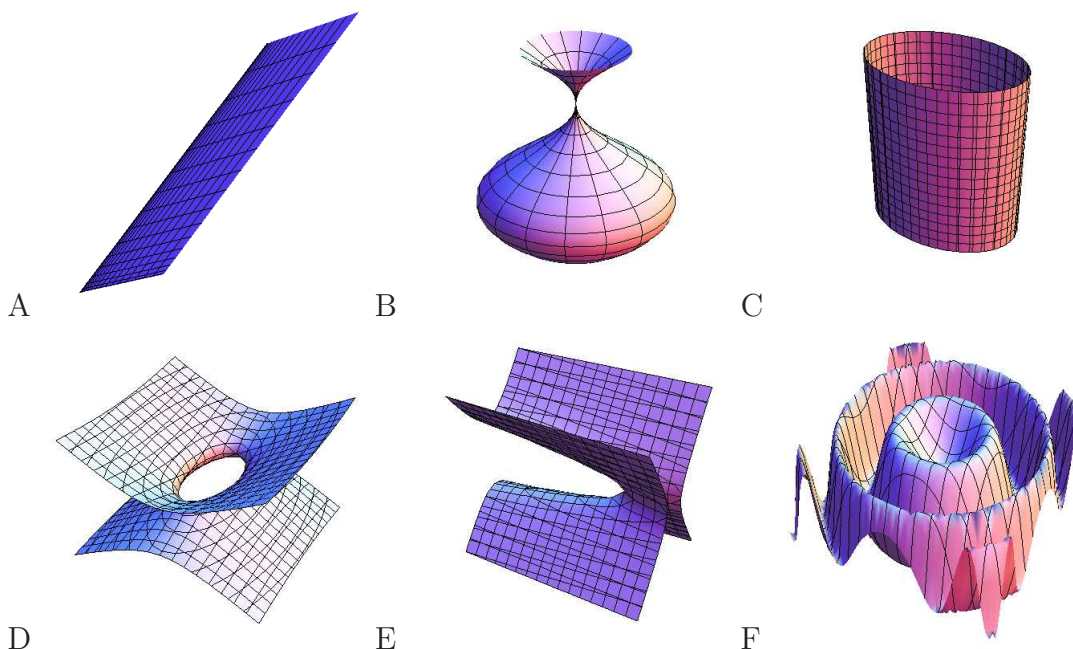
Enter 1-8 or 0 if no match	Object definition
	$\vec{r}(t) = [(2 + \cos(10t)) \cos(t), (2 + \cos(10t)) \sin(t), \sin(10t)]^T$
	$\vec{F}(x, y, z) = [-y, x, 2]^T$
	$\vec{r}(t, s) = [(2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s)]^T$
	$\{(x, y, z) \mid \sin(x^2) - \cos(y^2) = 1\}$
	$\vec{F}(x, y) = [x - y^2, y - x^2]^T$
	$xyz = 0$
	$x^2 + y^2 - z^2 = 1$
	$\{(x, y) \mid \sin(x^2 \sin(x))y + \sin(y - x) = c\}$
	$\vec{r}(t) = [\sin(t) + \cos(5t), \cos(t) + \cos(6t)]^T$

Problem 3) (10 points)

a) (4 points) Check every box to the left, for which the missing part to the right is $\nabla f(1, 2)$. The function $f(x, y)$ is an arbitrary nice function like for example $f(x, y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x, y) = f(1, 2)$ and has the property that $\vec{r}(0) = [1, 2]^T$.

Check	Topic	Statement
☐	Linearization	$L(x, y) = f(1, 2) + \boxed{} \cdot [x - 1, y - 2]^T$
☐	Chain rule	$\frac{d}{dt} f(\vec{r}(t)) _{t=0} = \boxed{} \cdot \vec{r}'(0)$
☐	Steepest descent	f decreases at $(1, 2)$ most in the direction of $\boxed{}$
☐	Estimation	$f(1 + 0.1, 1.99) \sim f(1, 2) + \boxed{} \cdot [0.1, -0.01]^T$
☐	Directional derivative	$D_{\vec{v}} f(1, 2) = \boxed{} \cdot \vec{v}$
☐	Level curve	of f through $(1, 2)$ has the form $\boxed{} \cdot [x - 1, y - 2]^T = 0$
☐	Vector projection	of $\nabla f(1, 2)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \boxed{})/\ \vec{v}\ ^2$
☐	Tangent line	of $\vec{r}(t)$ at $(1, 2)$ is parametrized by $\vec{R}(s) = [1, 2]^T + s \boxed{}$

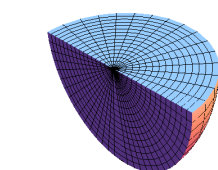
b) (3 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.



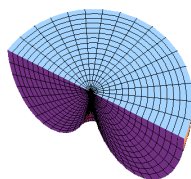
Enter A-F here	Function or parametrization
	$\vec{r}(u, v) = [u^2, v^2, u^2 + v^2]^T$
	$\vec{r}(u, v) = [(1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v), u]^T$
	$4x^2 + y^2 - 9z^2 = 1$
	$x - 9y^2 + 4z^2 = 1$
	$\vec{r}(u, v) = [u, v, \sin(u^2 + v^2)]^T$
	$4x^2 + 9y^2 = 1$

c) (3 points) Match the solids with the triple integrals:

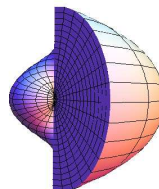
Enter A-D	3D integral computing volume
	$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/\cos(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{\sin(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^1 \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{2\pi} \int_0^{\pi} \int_0^{\theta} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$



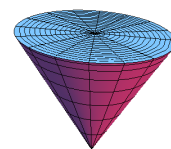
A



B



C



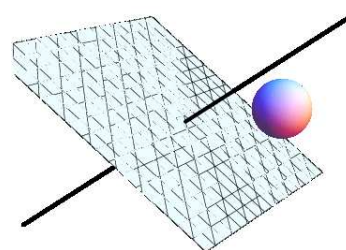
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Problem 4) (10 points)

We want to determine whether the distance of the sphere S of radius 1 centered at $P = (1, 2, 3)$ to the plane $E : x + y + z = 1$ is larger than the distance of the same sphere to the line $L : x + y = y + z = x + z$.

a) (5 points) Find the distance from the sphere S to the plane E .

b) (5 points) Find the distance from the sphere S to the line L .



Problem 5) (10 points)

Where does the vector field

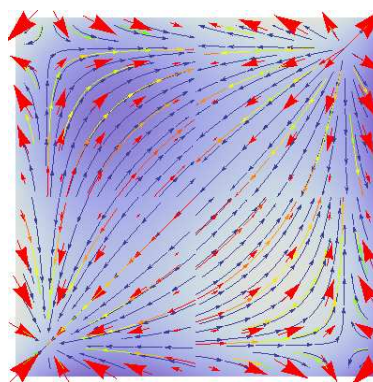
$$\vec{F}(x, y) = [P, Q]^T = [y(x^3 - 3x), x(y^3 - 3y)]^T$$

have maximal or minimal curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) .$$

a) (8 points) Find all extrema and determine whether they are maxima, minima or saddle points.

b) (2 points) Is there a global maximum of $f(x, y)$?



Problem 6) (10 points)

A sprinkler at position $(0,0,1)$ throws out water with constant speed and elevation angle 45 degrees. The water is under constant gravitational acceleration $[0,0,-10]^T$.

a) (5 points) Find the trajectory $\vec{r}(t)$, if the initial velocity is $\vec{r}'(t)|_{t=0} = [\cos(\theta), \sin(\theta), 1]^T$ and write down the formula for the arc length from $t = 0$ to $t = 1$. You do not have to start evaluating the integral.

b) (5 points) All the trajectories together form a surface $\vec{r}(\theta, t)$. Parametrize this surface and write down the formula for the surface area if $0 \leq t \leq 1$ and $0 \leq \theta \leq 2\pi$. You do not have to start evaluating the integral.



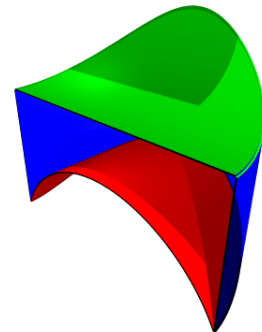
Problem 7) (10 points)

Compute the integral

$$\iiint_E x^2 dz dx dy ,$$

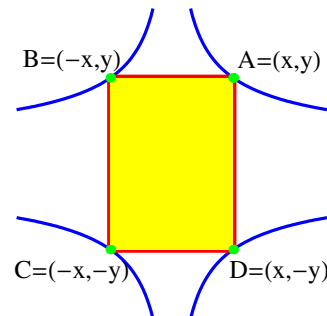
over the solid E defined by the four conditions

$$x^2 + y^2 \leq 1, y \geq 0, z < 4 - y^2, \text{ and } z > -5 + x^2 .$$



Problem 8) (10 points)

Use the method of Lagrange multipliers to find the centrally symmetric rectangle with corners $A = (x, y)$, $B = (-x, y)$, $C = (-x, -y)$, $D = (x, -y)$ on the curve $g(x, y) = x^2 y^4 = 1$ which has minimal circumference $f(x, y) = 4x + 4y$.



Problem 9) (10 points)

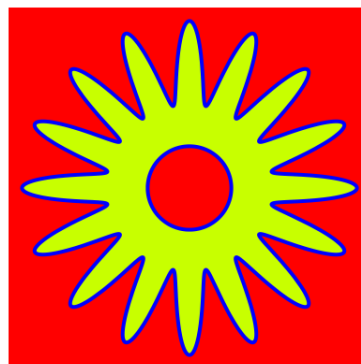
a) (5 points) Find the area of the region which is given in polar coordinates (r, θ) as

$$1 \leq r \leq 2 + \cos(16\theta) .$$

The picture of this region can be admired to the right.

b) (5 points) Find

$$\int_0^{\pi/2} \int_x^{\pi/2} \sin(y)/y \, dy dx .$$

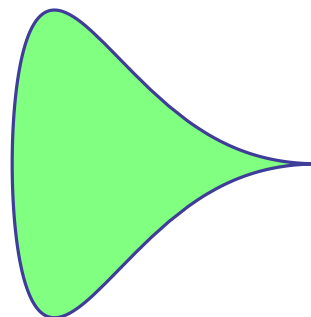


Problem 10) (10 points)

Find the area of the region enclosed by

$$\vec{r}(t) = [t^2, \frac{(\sin(\pi t))^2}{t}]^T$$

for $-1 \leq t \leq 1$. Use an integral theorem with a suitable vector field.



Problem 11) (10 points)

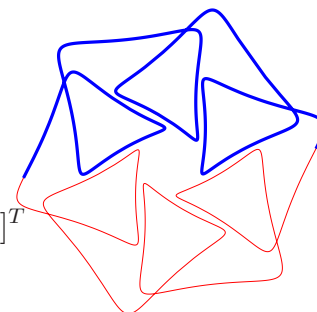
Compute the line integral of the vector field

$$\vec{F}(x, y) = [3x + 2xy^2, 2y + 2x^2y]^T$$

along the curve

$$\vec{r}(t) = [6 \cos(t) + 4 \cos(7t) + \sin(17t), 6 \sin(t) + 4 \sin(7t) + \cos(17t)]^T$$

from $t = 0$ to $t = \pi$.



Problem 12) (10 points)

Find the flux of vector field

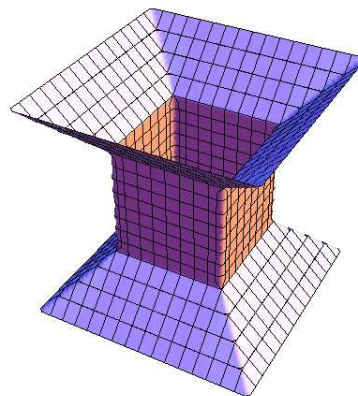
$$\vec{F}(x, y, z) = [y^2 \sin(z) - x, z^2 \cos(x^2), 5z]^T$$

through the surface S given by the $p \rightarrow \infty$ limit of

$$|x|^p + |y|^p - |z|^p = 1, \quad -2 < z < 2.$$

The surface is oriented that the normal vectors points outwards.

Hints. The surface (see picture) becomes closed if two not yet included square "lids" at $z = 2$ and $z = -2$ with corners at $(\pm 2, \pm 2, \pm 2)$ are added. You can use without proof that the volume of the solid is $(16 + 4)/2 + (16 + 4)/2 + 2 * 2 = 20 + 4 = 24$.



Problem 13) (10 points)

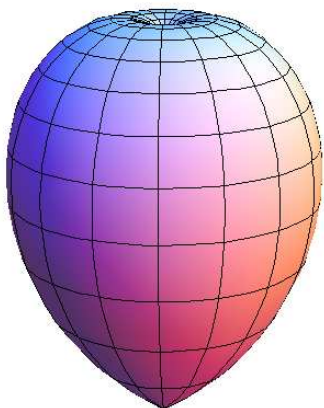
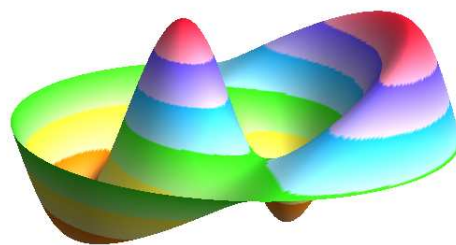
Compute the flux of the curl of the vector field

$$\vec{F}(x, y, z) = [x^2, y, \sin(z^2)^4]^T$$

through the surface which has the parametrization

$$\vec{r}(t, s) = [s \cos(t), s \sin(t), 3 \sin(s) \cos(t)]^T,$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$.



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x, y, z) = [z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y)]^T.$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field $\vec{F}$ through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.