1: Geometry and Distance

The geometry of Euclidean space like the **plane** $\mathbb{R}^2$ and **space** $\mathbb{R}^3$ remains a frontier: while we have explored the **micro-cosmos** with microscopes, the **macro-cosmos** with telescopes we only recently started to conquer the **meso-scale** with 3D scanning, 3D printing, mapping, computer simulations or using virtual and augmented reality frame works. Multivariable calculus is and will remain a key tool in all three scales.

> A point in the **plane** $\mathbb{R}^2$ has two **coordinates** $P = (x, y)$. A point in space $\mathbb{R}^3$ is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates define 4 **quadrants** in $\mathbb{R}^2$ or 8 **octants** in $\mathbb{R}^3$. These regions intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are bound by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In $\mathbb{R}^2$ its custom to orient the x-axis to the "east" and the y-axis "north". In $\mathbb{R}^3$, the most common coordinate system has the xy-plane as the "ground" and the z-coordinate axes pointing "up".

1 $P = (-2, -3)$ is in the third quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -8)$ is located on the negative z axis. The point $P = (1, 2, -3)$ is below the xy-plane. Can you spot the point Q on the xy-plane which is closest to P?

2 **Problem.** Find the midpoint M of $P = (7, 2, 5)$ and $Q = (-15, 4, 7)$. **Answer.** The midpoint is the average of each coordinate $M = (P + Q)/2 = (-4, 3, 6)$.

3 In computer graphics or photography, the xy-plane represents the **retina** or film plate and the z-coordinate measures the distance towards the viewer. In this **photographic coordinate** system, your eyes and chin define the plane $z = 0$ and the nose points in the positive z direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0)$, $(-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.

> The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

This is a **definition**, not a result! It is motivated by **Pythagoras theorem**, but we will **prove** the later result in a moment.

4 **Problem:** Find the distance $d(P, Q)$ between $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

Remarks.
1) A distance can be defined by taking any non-negative function $d(P, Q) = d(Q, P)$ which satisfies the **triangle inequality** $d(P, Q) + d(Q, R) \geq d(P, R)$ and which has the property that $d(P, Q) = 0$ if and only if $P = Q$. A set X equipped with such a distance function d is called a **metric space**.

Examples of distances on $\mathbb{R}^2$ are the **Manhattan (or taxi) distance** $d_m(P,Q) = |x-a|+|y-b|$, the **quartic distance** $d_4(P,Q) = ((x-a)^4+(y-b)^4$ or the **Fermat distance** $d_f(x,y) = d(x,y)$ if $y > 0$ and $d_f(x,y) = 1.33d(x,y)$ if $y < 0$. In the last example, the constant 1.33 is the **refractive index** in a model where the upper half plane is filled with air and the lower half plane with water. Shortest paths in this metric are broken lines: light rays get bent at the water surface. Each of these distances d, d_m, d_4, d_f equip the plane with a different metric.

2) Symmetry distinguishes the Euclidean distance from other distances. It is characterized by the property $d((1,0,0),(0,0,0)) = 1$ together with the requirement of rotational and translational and scaling symmetry $d(\lambda P, \lambda Q) = \lambda d(P,Q)$.

3) We usually work with a **right handed coordinate system**, where the x, y, z axes can be matched with the thumb, pointing and middle finger of the **right hand**. The photographers coordinate system is an example of a **left handed coordinate system**, where we use the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some fundamental laws in particle physics related to the weak force are different when observed in a mirror. Coordinate systems with different parity can not be rotated into each other.

4) When dealing with geometric problems in the plane, we leave the z-coordinate away and have $d(P,Q) = \sqrt{(x-a)^2+(y-b)^2}$, where $P = (x,y), Q = (a,b)$. Its important to work in $\mathbb{R}^2$ without referring to a possible $\mathbb{R}^3$ in which it might be embedded.

Points, **curves**, **surfaces** and **solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus next on spheres or circles.

A **circle** of radius $r \geq 0$ centered at $P = (a,b)$ is the collection of points in $\mathbb{R}^2$ which have distance r from P.

A **sphere** of radius ρ centered at $P = (a,b,c)$ is the collection of points in $\mathbb{R}^3$ which have distance $\rho \geq 0$ from P. The equation of a sphere is $(x-a)^2+(y-b)^2+(z-c)^2 = \rho^2$.

An **ellipse** is the collection of points P in $\mathbb{R}^2$ for which the sum $d(P,A)+d(P,B)$ of the distances to two points A, B is a fixed constant $l > 0$ larger than $d(A,B)$. This allows to draw the ellipse with a string of length l attached at A, B. When 0 is the midpoint of A, B, an algebraic description is the set of points which satisfy the equation $x^2/a^2 + y^2/b^2 = 1$.

5 **Problem:** Is the point $(3,4,5)$ outside or inside the sphere $(x-2)^2+(y-6)^2+(z-2)^2 = 16$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1+4+9}$. Since this is smaller than 4 the radius of the sphere, the point is inside.

6 **Problem:** Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1,0)$ and $B = (-1,0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2+36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a,0)$ satisfies this equation, $d(P,A) + d(P,B) = (a-1)+(a+1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0,b)$ satisfying it gives $d(Q,A)+d(P,B) = 2\sqrt{b^2+1} = 3$ or $b = \sqrt{5}/2$.

When **completing the square** of an equation $x^2+bx+c=0$, we add $(b/2)^2-c$ on both sides of the equation in order to get $(x+b/2)^2=(b/2)^2-c$. Solving for x gives $x=-b/2\pm\sqrt{(b/2)^2-c}$.

7 The equation $2x^2-10x+12=0$ is equivalent to $x^2+5x=-6$. Adding $(5/2)^2$ on both sides gives $(x+5/2)^2=1/4$ so that $x=2$ or $x=3$.

8 The equation $x^2+5x+y^2-2y+z^2=-1$ is after completion of the square $(x+5/2)^2-25/4+(y-1)^2-1+z^2=-1$ or $(x-5/2)^2+(y-1)^2+z^2=(5/2)^2$. We see a sphere **center** $(5/2,1,0)$ and **radius** $5/2$.

The method is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [1] In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. But ideas do not grow in a vacuum; Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. [2]

A point in $\mathbb{R}^4$ is labeled with four coordinates (t,x,y,z). This is also the space $\mathbb{H}$ of **Quaternions** for which one has nice arithmetic. In how many regions do the coordinate hyperplanes $t=0,x=0,y=0,z=0$ cut this space? Answer: There are 16 hyper-regions and each of them contains one of the 16 points (x,y,z,w), where x,y,z,w are either $+1$ or -1.

Homework

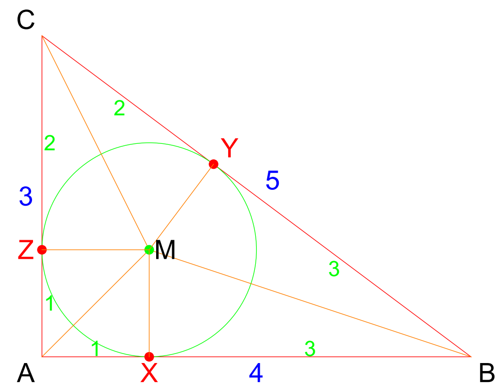
1 Describe in a each case in words the set of points $P=(x,y,z)$ in $\mathbf{R}^3$ that are represented by
a) $(y+3)^2+(z-19)^2-81=0$ c) $x^4z^2y^8=0$
b) $3x-4y+9z=36$ d) $x^2=y^2-8$

2 a) Find the distance of $P=(18,12,5)$ to the x-axes.
b) Find the distance of $P=(2,13,-8)$ to the yz-plane.
c) Find the center of the sphere $4x^2+4y^2-320y+4(z+2)^2=144$.
d) What is the xz-trace of the sphere in c)?

[1] The book "The mathematics of Egypt, Mesopotamia, China, India and Islam, by Ed Victor Katz, page 542 contains translations of some of this work.

[2] An entertaining read is "Descartes secret notebook" by Amir Aczel which deals with an other discovery of Descartes. By the way, Descartes also formulated first a Goldbach conjecture.

3 Verify that the radius of the inscribed circle in a $3:4:5$ triangle is 1. Use the picture with the triangle ABC given by $A=(0,0), B=(4,0), C=(0,3)$, introduce $M=(1,1)$ then get the coordinates of the points X, Y, Z, then compute the distances.



4 You play billiard in the table $\{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq 8\}$. a) Hit the ball at $(3,2)$ to reach the hole $(4,8)$ bouncing 3 times at the left wall and three times at the right wall and no other walls. Find the length of the shot.

b) Hit from $(3,2)$ to reach the hole $(4,0)$ after hitting twice the left and twice the right wall as well as the top wall $y=8$ once. What is the length of the trajectory?

5 A point $P=(a,b,c,d)$ in $\mathbb{H}=\mathbb{R}^4$ for which all coordinates are integers is called a **Lipschitz quaternion**. Define the **norm** $N(P)=a^2+b^2+c^2+d^2$. A point P with prime norm is called a **Lipschitz prime**.

a) Lagrange's four square theorem assures that every integer n appears as the norm of some Quaternion. Find a Lipschitz prime with norm $p=211$.

b) Define $P*Q=(a,b,c,d)*(p,q,r,s)=(ap-bq-cr-ds, bp+aq-dr+cs, cp+dq+ar-bs, dp-cq+br+as)$. Verify that $N(PQ)=N(P)N(Q)$. (This is a bit of a hike but you can hone your basic algebra skills).

2: Vectors and Dot Product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$ which we simply write also as $[x-a, y-b, z-c]^T$ or $\langle x-a, y-b, z-c\rangle$ to save space. It is custom in linear algebra to call $[3, 4, 5]$ a row vector and its transpose $[3, 4, 5]^T = \langle 3, 4, 5\rangle$ a **column vector**. As it goes from P to Q we write $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in $\vec{v} = [p, q, r]^T$ are called the **components** of $\vec{v}$.

Vectors can be drawn **anywhere** in space. But two vectors with the same components are considered **equal**. Vectors can be translated into each other if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]^T$ heads to the point (p, q, r). One can therefore identify points $P = (a, b, c)$ with vectors $\vec{v} = [a, b, c]^T$ attached to the origin. For clarity, we often draw an arrow on top of vectors and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to write $[2, 3, 4]^T$ or $\langle 2, 3, 4\rangle$ for vectors and $(2, 3, 4)$ for points.

The **sum** of two vectors is $\vec{u} + \vec{v} = [u_1, u_2]^T + [v_1, v_2]^T = [u_1 + v_1, u_2 + v_2]^T$. The **scalar multiple** is $\lambda\vec{u} = \lambda[u_1, u_2]^T = [\lambda u_1, \lambda u_2]^T$. The **difference** $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

On can check commutativity, associativity, or distributivity rules for vectors as for numbers.

The vectors $\vec{i} = [1, 0]^T$, $\vec{j} = [0, 1]^T$ are called the **standard basis vectors** in the plane. In space, one has the standard basis vectors $\vec{i} = [1, 0, 0]^T$, $\vec{j} = [0, 1, 0]^T$, $\vec{k} = [0, 0, 1]^T$.

Every vector $\vec{v} = [p, q]^T$ in the plane can be written as a combination $\vec{v} = p\vec{i} + q\vec{j}$ of standard basis vectors. Every vector $\vec{v} = [p, q, r]^T$ in space can be written as $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$. Vectors appear everywhere in applications. For example in mechanics: if $\vec{r}(t) = [f(t), g(t)]^T$ is a point in the plane which depends on time t, then $\vec{v} = [f'(t), g'(t)]^T$ is the **velocity vector** at $\vec{r}(t)$. Here $f'(t), g'(t)$ are the derivatives. In physics, where we often want to determine forces acting on objects, these forces are represented as vectors. In particular, electromagnetic or gravitational fields or velocity fields in fluids are described by vectors. Vectors appear also in computer science: the "scalable vector graphics format" is a standard for the web for describing graphics. In quantum computation, rather than working with bits, one deals with **qbits**, which are vectors. Finally, **color** can be written as a vector $\vec{v} = [r, g, b]^T$, where r is **red**, g is **green** and b is **blue** component of the color vector. An other coordinate system for color is $\vec{v} = [c, m, y]^T = [1-r, 1-g, 1-b]^T$, where c is **cyan**, m is **magenta** and y is **yellow**. Vectors appear also in probability theory and statistics. On a finite probability space for example, a **random variable** is nothing else than a vector.

The addition and scalar multiplication of vectors satisfy the laws known from **arithmetic**. They are **commutativity** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, **associativity** $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$, $r*(s*\vec{v}) = (r*s)*\vec{v}$ as well as **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$ and $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$, where $*$ denotes multiplication with a scalar.

The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P,Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is called a **unit vector**.

1 $|[3,4]^T| = 5$ and $|[3,4,12]^T| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $[3/5,4/5]^T$ and $[3/13,4/13,12/13]^T$. The only vector of length 0 is the **zero vector** $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = [a,b,c]^T$ and $\vec{w} = [p,q,r]^T$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

Remarks.
1) Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, the **Dirac notation** $[\vec{v}|\vec{w}]^T$ is used in quantum mechanics or the **Einstein notation** $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity. The dot product is also called **scalar product** or **inner product**.
2) Any product $g(v,w)$ which is linear in v and w and satisfies the symmetry $g(v,w) = g(w,v)$ and $g(v,v) \geq 0$ and $g(v,v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v,w) = 3v_1w_1 + 2v_2w_2 + v_3w_3$.

The dot product determines distances and distances determines the dot product.

Proof: Write $v = \vec{v}$. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v+w) \cdot (v+w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

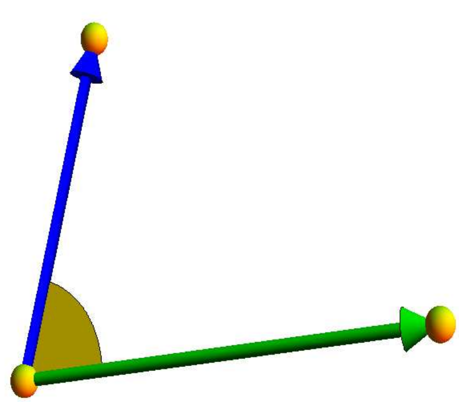
$$v \cdot w = (|v+w|^2 - |v|^2 - |w|^2)/2 \ .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. If $|w| = 0$, there is nothing to show. Otherwise, assume $|w| = 1$ by dividing the equation by $|w|$. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw) \cdot (v - aw)$ to get $0 \leq (v - (v \cdot w)w) \cdot (v - (v \cdot w)w) = |v|^2 + (v \cdot w)^2 - 2(v \cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$.

Having established this, it is possible to give a definition of what an **angle** is, without referring to any geometric pictures:

The **angle** between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0,\pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Since cos maps $[0,\pi]$ in a 1:1 manner to $[-1,1]$, this is well defined.



Al Kashi's theorem: If a, b, c are the side lengths of a triangle ABC and α is the angle at the vertex C, then $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The angle **definition** works in any space with a dot product. In **statistics** one works with vectors of n components. They are called **data** or random variables and $\cos(\alpha)$ is called the **correlation** between two random variables $\vec{v}, \vec{w}$ of zero **expectation** $\mathrm{E}[\vec{v}] = (v_1 + \cdots + v_n)/n$. The dot product $(v_1 w_1 + \cdots + v_n w_n)/n$ is then the **covariance**, the scaled length $|v|/\sqrt{n}$ is the **standard deviation** and denoted by $\sigma(v)$. The formula $\mathrm{Corr}[v, w] = \mathrm{Cov}[v, w]/(\sigma(v)\sigma(w))$ for the correlation is the familiar angle formula. Statistics shows that geometry in arbitrary dimensions can be useful.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]^T$ is orthogonal to $\vec{w} = [-3, 2]^T$.

We can now prove the **Pythagoras theorem**:

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

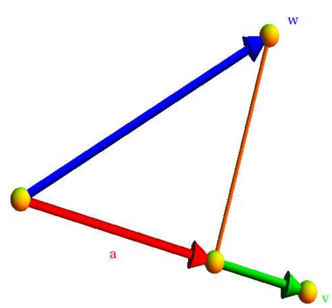
Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

Remarks:
1) We have seen how results of Pythagoras (570-495 BC) and Al Khashi (1380-1429) can be **derived from scratch** on a space V equipped with a dot product. The dot product appeared much later in mathematics (Hamilton 1843, Grassman 1844, Sylvester 1851, Cayley 1858), first using quaternions. All modern textbook essentially follow Gibbs, who got rid of quaternions and focused on dot and cross product instead. While we have used geometry as an intuition, the structure was built algebraically without any further assumptions. This is mathematics: if we have a space V in which addition $\vec{v} + \vec{w}$ and scalar multiplication $\lambda\vec{v}$ is given and in which a dot product is defined, then all the results we have seen apply. We have **not built on** results of Al Khashi or Pythagoras but we have **derived and proven** them. Additionally we obtained a clear **definition** what length and angle is.
2) This derivation works in any dimension. Why do we care about higher dimensions? A compelling motivation is **statistics**. Given 12 data points like the average monthly temperatures in a year, we deal with a 12-dimensional space. Geometry is useful to describe data. Pythagoras theorem is reformulated as the property that the variance of two uncorrelated random variables adds up with the formula $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3) A far reaching generalization of Euclidean geometry is obtained if the dot product $g(v, w)$ can depend on location. This produces **Riemannian geometry**. It allows to work with spaces which are intrinsically **curved**. This geometry is important in **general relativity** which describes gravity in a geometric way and which is one of the pillars of modern physics. But it appears in daily life too. If you look close at an object on a hot asphalt street, the object can appear distorted

or flickers. The dot product and so the angles depends on the temperature of the air. Light rays no more move on straight lines but is bent. In extreme cases, when the curvature of light rays is larger than the curvature of the earth, it leads to **Fata morgana** effects: one can see objects which are located beyond the horizon.

4) Why don't we just define vectors as algebraic objects $[1, 2, 3]^T$ without attaching them to points in space? The reason is that in applications of physics or geometry, we want to work with **affine vectors**, vectors for which the base point is attached somewhere. **Forces** for example act on points of a body, vector fields are families of vectors attached to points of space. Considering vectors with the same components as "equal" gives then the **vector space** in which we do the algebra. One could define a vector space axiomatically and then build from this affine vectors, but almost nothing is gained from this abstraction. An even more modern point of view replaces affine vectors with members of a **tangent bundle** a geometry where at each point a tangent space is attached. Such a generalization is needed when dealing with spaces which are not flat. Even more general is to allow the attached space to be a more general space like a "group" called fibres. Such "fibre bundles" are the framework of mathematical concepts which describe elementary particles or even space itself. Attaching a circle for example at each point leads to electromagnetism, attaching classes of two dimensional matrices leads to the weak force and attaching certain three dimensional matrices leads to the strong force. Allowing this to happen in a curved framework incorporates gravity. One of the unresolved challenges is to include quantum mechanics into that picture. Fundamental physics contains the basic question: "What is space"?

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.



2 For example, with $\vec{v} = [0, -1, 1]^T$, $\vec{w} = [1, -1, 0]^T$, $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]^T$. Its length is $1/\sqrt{2}$.

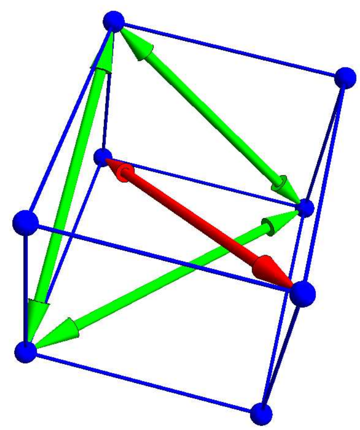
3 Projections are important in physics. If a wind force $\vec{F}$ affects a car driving in the direction $\vec{w}$ and P denotes the projection onto $\vec{w}$ then $P_{\vec{w}}(\vec{F})$ is the force which accelerates or slows down the car.

The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Homework

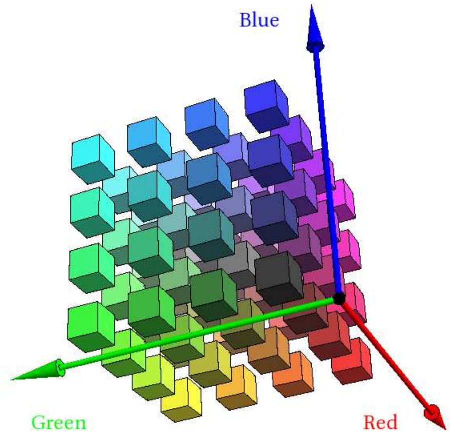
1 Find a **unit vector** parallel to $\vec{u} + 2\vec{v} + 4\vec{w}$ if $\vec{u} = [-10, 2, 9]^T$ and $\vec{v} = [1, 1, 3]^T$ and $\vec{w} = [3, 1, 1]^T$.

2 An **Euler brick** is a **cuboid** with side lengths a, b, c such that all face diagonals are integers.
a) Verify that $\vec{v} = [a, b, c]^T = [275, 252, 240]^T$ is a vector which leads to an Euler brick. Halcke found the first one in 1719.
b) (*) Verify that $[a, b, c]^T = [u(4v^2 - w^2), v(4u^2 - w^2), 4uvw]^T$ leads to an Euler brick if $u^2 + v^2 = w^2$.
(Sounderson 1740) If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

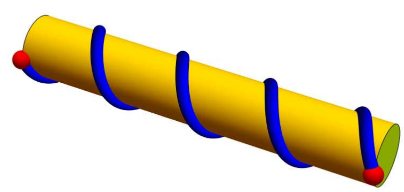


3 **Colors** are encoded by vectors $\vec{v} = [$red, green, blue$]^T$. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.
a) Determine the angle between the colors yellow and magenta.
b) What is the vector projection of the magenta-orange mixture $\vec{x} = (\vec{v} + \vec{w})/2$ onto green $\vec{y}$?

(0,0,0)	black	(0,0,1)	blue
(1,1,1)	white	(1,1,0)	yellow
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	gray	(1,0,1)	magenta
(1,0,0)	red	(0,1,1)	cyan
(0,1,0)	green	$(1,\frac{1}{2},0)$	orange
$(0,1,\frac{1}{2})$	vivid	$(1,1,\frac{1}{2})$	khaki
$(1,\frac{1}{2},\frac{1}{2})$	pink	$(\frac{1}{2},\frac{1}{4},0)$	brown



4 A rope is wound exactly 5 times around a stick of circumference 1 and length 12. How long is the rope?



5 a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.
b) Find the vector projection of the main diagonal $\vec{v} = [1,1,1]^T$ onto the side diagonal $\vec{w} = [1,1,0]^T$.
c) Find the scalar projection of $\vec{v}$ on $\vec{w}$.

3: Cross product

The **cross product** of two vectors $\vec{v} = [v_1, v_2]^T$ and $\vec{w} = [w_1, w_2]^T$ in the plane is the **scalar** $v_1 w_2 - v_2 w_1$.

To remember this, write it as a determinant of a matrix A which is a 2×2 array of the numbers). $\det(A)$ is the product of the diagonal entries minus the product of the side diagonal entries. $\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]^T$ and $\vec{w} = [w_1, w_2, w_3]^T$ in space is defined as the **vector**

$$\vec{v} \times \vec{w} = [v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1]^T .$$

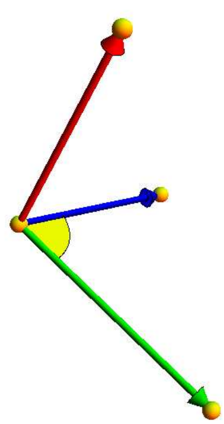
To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$.

1 The cross product of $[1, 2]^T$ and $[4, 5]^T$ is $5 - 8 = -3$.

2 The cross product of $[1, 2, 3]^T$ and $[4, 5, 1]^T$ is $[-13, 11, -3]^T$.



In space, the cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. The product is anti-commutative.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.

Length formula for the cross product: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$.

Proof: verify first the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ which is also called **Cauchy-Binet** formula by direct computation (done in class). Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

Note that we have given the **definition** of area, so that nothing needs to be proven. To see that the definition fits with our common intuition we have about area, note that $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. The area formula proves the sin-formula because the area does not depend on which pair of sides to a triangle we take. The area makes sense because it is linear in each of the vectors $\vec{v}$ and $\vec{w}$: scale one by a factor $\lambda = 2$ for example, doubles the area.

$\vec{v} \times \vec{w}$ is zero if and only if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda\vec{w}$ for some real λ.

Proof. Use the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are considered parallel even so sometimes the notion **anti-parallel** is used.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

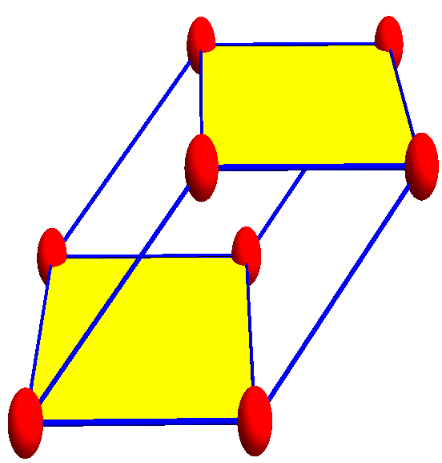
Proof. Express the area of the triangle in three different ways:

$$ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta) .$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

3 If $\vec{v} = [a, 0, 0]^T$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]^T$, then $\vec{v} \times \vec{w} = [0, 0, ab\sin(\alpha)]^T$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.



The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.

The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors define a **right-handed** coordinate system.

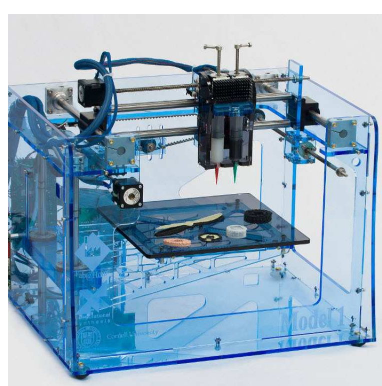
Again, there was no need to prove anything because we **defined** volume and orientation. Why does this fits with our intuition? The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector, we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with twice the volume.

4 **Problem:** You have two apples of the same shape, but one has a 3 times larger diameter. What is their weight ratio?
Answer. For a parallelepiped spanned by $[a, 0, 0]^T$ $[0, b, 0]^T$ and $[0, 0, c]^T$, the volume is the triple scalar product abc. If a, b, c are all tripled, the volume gets multiplied by a factor 27. Now cut each apple into the same amount of parallelepipeds, the larger one with slices 3 times as large too. Since each of the pieces has 27 times the volume, also the apple is 27 times heavier!

5 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that the solid is spanned by the vectors $\vec{u} = [1, 2, 1]^T$, $\vec{v} = [3, 2, 1]^T$, and $\vec{w} = [0, 3, 1]^T$. We get $\vec{v} \times \vec{w} = [-1, -3, 9]^T$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

6 **Problem.** A **3D scanner** is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0, 1, 1), Q = (1, 1, 0)$ and $R = (1, 2, 3)$. Find the area of the triangle. **Solution.** We have to find the length of the cross product of $\vec{PQ}$ and $\vec{PR}$ which is $[1, -3, 1]^T$. The length is $\sqrt{11}$.



7 **Problem.** The scanner detects an other point $A = (1, 1, 1)$. On which side of the triangle is it located if the cross product of $\vec{PQ}$ and $\vec{PR}$ is considered the direction "up". **Solution.** The cross product is $\vec{n} = [1, -3, 1]^T$. We have to see whether the vector $\vec{PA} = [1, 0, 0]^T$ points into the direction of $\vec{n}$ or not. To see that, we have to form the dot product. It is 1 so that indeed, A is "above" the triangle. Note that a triangle in space a priori does not have an orientation. We have to tell, what direction is "up". That is the reason that file formats for 3D printing like contain the data for three points in space as well as a vector, telling the direction.

Homework

1 a) Find a unit vector perpendicular to the space diagonal $[1, 1, 1]^T$ and the face diagonal $[1, 1, 0]^T$ of the cube.
b) Find the volume of the parallelepiped for which the base parallelogram is given by the points $P = (5, 2, 2), Q = (3, 1, 2), R = (1, 4, 2), S = (-1, 3, 2)$ and which has an edge connecting P with $T = (5, 6, 8)$.
c) Find the area of the base and use b) to get the height of the parallelepiped.

2 a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.

3 To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $[a, b, c]^T$ normal to the plane and fix d so that P is in the plane.

4 Verify the "BAC minus CAB" formula (due to Lagrange) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space.

5 A product $*$ is said to satisfy the **cancellation property** if for all $x, y, z \neq 0$: $x * z = y * z$ implies that $x = y$.
a) Does the dot product satisfy the cancellation property?
b) Does the cross product satisfy the cancellation property?

4: Lines and Planes

A point $P = (p, q, r)$ and a vector $\vec{v} = [a, b, c]^T$ define the **line**

$$L = \{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ t \in \mathbb{R} \} .$$

The line consists of all points obtained by adding a multiple of the vector $\vec{v} = [a, b, c]^T$ to the vector $\vec{OP} = [p, q, r]^T$. The line contains the point P as well as a copy of $\vec{v}$ attached to P. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". At time $t = 0$, we are at the point P, whereas at time $t = 1$ we are at $\vec{OP} + \vec{v}$.

If t is restricted to values in a **parameter interval** $[s, u]$, then $L = \{[p, q, r]^T + t[a, b, c]^T, \ s \leq t \leq u \}$ is a **line segment** which connects $\vec{r}(s)$ with $\vec{r}(u)$.

1 To get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, form the vector $\vec{v} = \vec{PQ} = [1, 3, 4]^T$ and get $L = \{[x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T; \}$. This can be written also as $\vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t]^T$. If we write $[x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for t:

$$L = \{(x, y, z) \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4 \} .$$

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]^T$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$. The interpretation is that the line is written as an intersection of two planes.

2 Find the symmetric equations for the line through the points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$, first form the parametric equations $[x, y, z]^T = [0, 1, 1]^T + t[2, 2, 3]^T$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for t gives the symmetric equation $x/2 = (y - 1)/2 = (z - 1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.

4 An example is $\Sigma = \{[x, y, z]^T = [1, 1, 2]^T + t[2, 4, 6]^T + s[1, 0, -1]^T\ \}$. This is called the **parametric description** of a plane.

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = [a, b, c]^T$, this means $ax + by + cz = ax_0 + by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have shown:

The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point P is

$$ax + by + cz = d\ ,$$

where $[a, b, c]^T = \vec{v} \times \vec{w}$ and d is obtained by plugging in P.

5 **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = [1, 2, 0]^T$ and $\vec{w} = [2, 2, 2]^T$. We have $\vec{n} = [4, -2, -2]^T$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two normal vectors $\vec{n} = [a, b, c]^T$ and $\vec{m} = [e, f, g]^T$.

6 **Problem:** Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. **Answer:** find the angle between $\vec{n} = [1, 1, 0]^T$ and $\vec{m} = [1, 1, 1]^T$. It is $\arccos(2/\sqrt{6})$.

Finally, lets look at some distance formulas.

1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|[-7, 0, -3]^T \cdot [2, 4, 5]^T|}{|[2, 4, 5]^T|} = \frac{29}{\sqrt{45}}\ .$$

2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2,3,1)$ to the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$, compute

$$d(P,L) = \frac{|[-1,-2,1]^T \times [5,0,1]^T|}{[5,0,1]^T} = \frac{|[-2,6,10]^T|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}} .$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$ form the cross product of $[-1,1,0]^T$ and $[5,1,2]^T$ is $[2,2,-6]^T$. The distance between these two lines is

$$d(L,M) = \frac{|(3,1,2) \cdot (2,2,-6)|}{|[2,2,-6]^T|} = \frac{4}{\sqrt{44}} .$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma,\Pi) = \frac{|e-d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

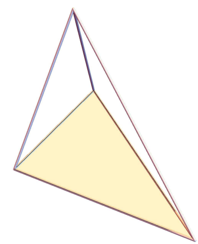
$$\frac{|8-1|}{|[5,4,3]^T|} = \frac{7}{\sqrt{50}} .$$

The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites. Each GPS satellite sends periodically signals which are triggered by an atomic clock. While the distance to each satellite is not known, the **difference** from the distances to two satellites can be determined from the time delay of the two signals. With this clever trick, the receiver does not need to contain an atomic clock itself. To understand this better, we need to know about functions of three variables and surfaces. This will be the topic of next week and also get us started with calculus.

Homework

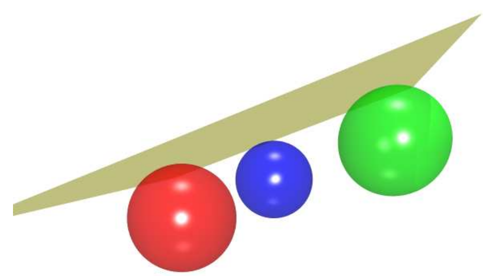
1 Given the three points $P = (7, 4, 5)$ and $Q = (1, 3, 9)$ and $R = (4, 2, 10)$. find the parametric and symmetric equation for the line perpendicular to the triangle PQR passing through its center of mass $(P + Q + R)/3$.

2 A regular tetrahedron has vertices at the points $P_1 = (0, 0, 6)$, $P_2 = (0, \sqrt{32}, -2)$, $P_3 = (-\sqrt{24}, -\sqrt{8}, -2)$ and $P_4 = (\sqrt{24}, -\sqrt{8}, -2)$. Find the distance between two edges which do not intersect.



3 Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + 4t, y = 1 - 4t, z = 8t$ and intersects this line in a point Q.

4 Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax+by+cz = d$ which touches all of three spheres from the same side.



5 a) Find the distance between the point $P = (3, 3, 4)$ and the line $2x = 2y = 2z$.
b) Parametrize the line $\vec{r}(t) = [x(t), y(t), z(t)]^T$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify that the minimal value agrees with a).