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- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1,2 and 6, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

- 1)  T  F The surface given in spherical coordinates as  $z^2 = 1 + \rho^2 \cos^2(\phi)$  is a double cone.

**Solution:**

This means  $z^2 - x^2 - y^2 = 1$  which is a two sheeted hyperboloid

- 2)  T  F If  $\vec{r}(t)$  is a curve, then  $\int_a^b |\vec{r}'(t)| dt \geq |\int_a^b \vec{r}'(t) dt|$

**Solution:**

The right hand side is direct connection.

- 3)  T  F The vector projection of the unit tangent vector  $\vec{T}$  onto the normal vector is  $\text{Proj}_{\vec{N}}(\vec{T}) = \vec{0}$ .

**Solution:**

Because they are perpendicular.

- 4)  T  F The acceleration of  $\vec{r}(t) = [t^3, t^2, t]^T$  at time  $t = 0$  is  $[0, 2, 0]^T$ .

**Solution:**

It is  $[0, 2, 0]^T$ .

- 5)  T  F If  $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$  then  $\vec{v}, \vec{w}$  are parallel.

**Solution:**

The first statement means that the cross product is zero.

- 6)  T  F The vector  $[3, 3, 0]^T$  is perpendicular to the plane  $x - y + z = 1$ .

**Solution:**

Indeed, it is parallel to the normal vector.

- 7)  T  F The point  $(1, 1, 0)$  has the cylindrical coordinates  $(r, \theta, z) = (\sqrt{2}, \pi/2, 0)$ .

**Solution:**

Check the angles and length

- 8)  T  F The parametrized curve  $\vec{r}(t) = [3, 4 + 3 \cos(t), 2 + 4 \sin(t)]^T$  is an ellipse in space.

**Solution:**

Indeed, it is part of the  $yz$ -plane.

- 9)  T  F The curvature of the curve  $\vec{r}(t) = [t^2, 2t^2, 3t^2]^T$  is zero everywhere.

**Solution:**

The curve is a line.

- 10)  T  F There are two nonzero vectors  $\vec{v}$  and  $\vec{w}$  such that  $|\vec{v} \times \vec{w}| = 2|\vec{v}||\vec{w}|$ .

**Solution:**

It would be possible if the vectors could be length 1. IN general it is not.

- 11)  T  F If the dot product between two unit vectors is  $-1$ , then the two vectors are parallel.

**Solution:**

Yes,  $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha) = \cos(\alpha) = -1$  shows that  $\alpha = \pi$ .

- 12)  T  F It is possible that  $|(\vec{u} \times \vec{v}) \times \vec{w}| = 1$  for unit vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ .

**Solution:**

Take  $\vec{u} = [1, 0, 0]^T$ ,  $\vec{v} = [0, 1, 0]^T$  and  $\vec{w} = [0, 1, 0]^T$ .

- 13)  T  F If the velocity and the acceleration is parallel at all times, then the curvature of a curve  $\vec{r}(t)$  is zero for all  $t$ .

**Solution:**

Use the formula for curvature.

- 14)  T  F The arc length of the curve  $[t, t^2]^T$  from  $t = 0$  to  $t = 1$  is larger than  $\sqrt{2}$ .

**Solution:**

It is a parabola from  $(0, 0)$  to  $(1, 1)$ .

- 15)  T  F Given three lines in space, then at least two of them are parallel or two of them intersect.

**Solution:**

No, can all be skew. Take the coordinate axes and move them around.

- 16)  T  F The curve  $\vec{r}(t) = [\cos(t^2), \sin(t^2), t^3]^T$  is located on a cylinder.

**Solution:**

$$x^2 + y^2 = 1$$

- 17)  T  F If  $f(x, y) = x^8 y^9 \sin(x^5 y^7)$ , then  $f_{xxyx} = f_{xyxx}$ .

**Solution:**

This is always true.

- 18)  T  F The surface parametrized by  $\vec{r}(u, v) = [u^3 + v^3, u - v, u^3]^T$  is a plane.
- 19)  T  F The point given in spherical coordinates as  $(\rho, \theta, \phi) = (1, \pi, \pi/2)$  corresponds to the point  $(x, y, z) = (-1, 0, 0)$ .
- 20)  T  F  $\vec{w} \times (\vec{u} \times \vec{w}) = \vec{0}$  for all vectors  $\vec{u}, \vec{w}$ .

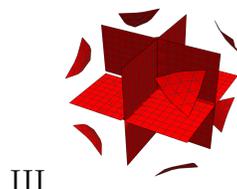
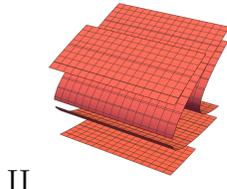
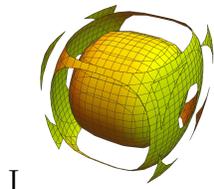
**Solution:**

A counter example is  $u = i, w = j$ .

Total

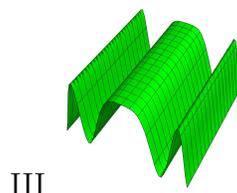
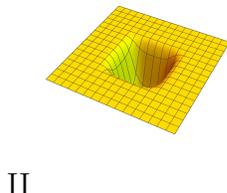
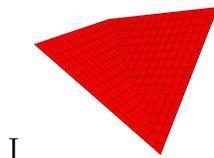
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour plots. Enter O, if there is no match.



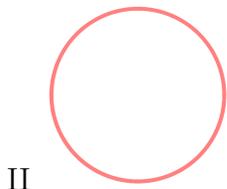
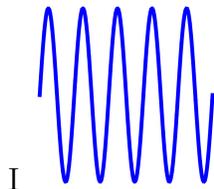
Function $g(x, y, z) =$	Enter O,I,II or III
$y + z$	
$x^4 + y^2 + z^2$	
$xyz$	
$z^2 - y$	
$x^4 + z^4$	

b) (2 points) Match the graphs of the functions  $f(x, y)$ . Enter O, if there is no match.



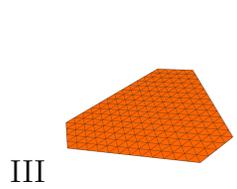
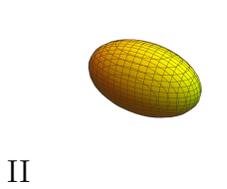
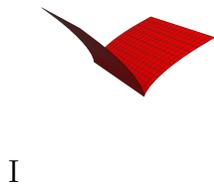
Function $f(x, y) =$	Enter O,I,II or III
$\cos(x^2)$	
$ x + y $	
$-\exp(-x^4 - y^4)$	
$x^3$	
$1/(1 + x^2)$	

c) (2 points) Match the plane curves with their parametrizations  $\vec{r}(t)$ . Enter O, if there is no match.



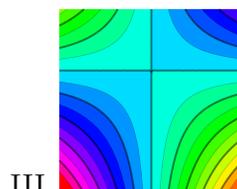
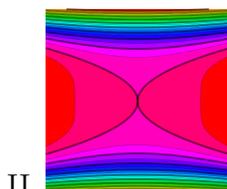
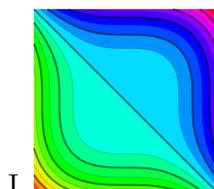
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = [t, \sin(5t)]^T$	
$\vec{r}(t) = [\sin(5t), \sin(t)]^T$	
$\vec{r}(t) = [\sin(t), \sin(5t)]^T$	
$\vec{r}(t) = [\sin(5t), \cos(5t)]^T$	
$\vec{r}(t) = [\sin(5t), t]^T$	

d) (2 points) Match functions  $g$  with level surface  $g(x, y, z) = 1$ . Enter O, if no match.



Function $g(x, y, z) =$	Enter O, I,II or III
$x^2 - y^2 + z^2 = 1$	
$x - y - z = 1$	
$xy + yz - x^2z = 1$	
$y^3 = z^2$	
$x^2/4 + y^2 + z^2/2 = 1$	

e) (2 points) Match the contour maps to a function  $f(x, y)$ . Enter O if no match.



$f(x, y) =$	Enter O,I,II or III
$x^2 - y^4$	
$xy - x$	
$x$	
$y$	
$x^3 + y^3$	

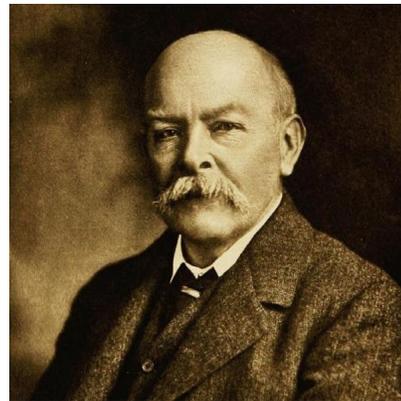
**Solution:**

- a) O,I,III,II,0
- b) III,I,II,0,0
- c) I,0,0,II,III
- d) O,III,O,I,II
- e) II,III,O,0,I

Problem 3) (10 points)

Let  $\vec{E} = [1, 1, 1]^T$  be a vector representing an electric field and  $\vec{B} = [1, 2, 3]^T$  a vector representing a magnetic field. Finally let  $\vec{v} = [0, 0, 1]^T$  be a velocity vector. Compute the:

- a) (2 points) Poynting vector  $\vec{S} = \vec{E} \times \vec{B}$ .
- b) (2 points) Electromagnetic energy density  $|\vec{E}|^2 + |\vec{B}|^2$ .
- c) (2 points) Unit vector in the direction of  $\vec{B}$ .
- d) (2 points) Lorentz force  $\vec{E} + \vec{v} \times \vec{B}$ .
- e) (2 points) Electric power  $\vec{E} \cdot \vec{v}$ .



The Poynting vector is used when designing coaxial cables used in studio equipment, antennas or television cable. It is named after **Henry Poynting** (1852-1914) seen above.

a) Answer =

b) Answer =

c) Answer =

d) Answer =

e) Answer =

### Solution:

- a)  $[1, -2, 1]^T$ .
- b) 17
- c)  $[1, 2, 3]^T / \sqrt{14}$
- d)  $[-1, 2, 1]^T$
- e) 1

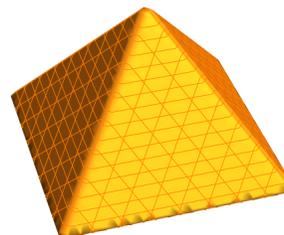
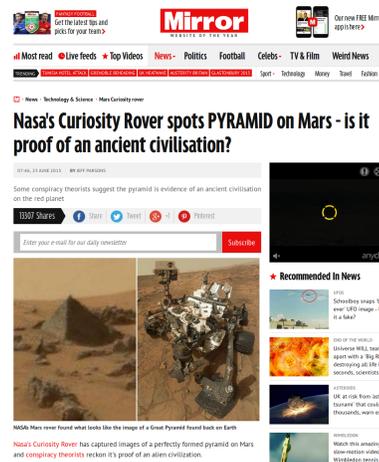
### Problem 4) (10 points)

On June 23, 2015, the curiosity rover spotted a 'Pyramid' on mars. Alien-hunters got excited, some claiming even it is proof of an alien civilization. The pyramid has the shape

$$|x| + |y| + |z| \leq 1, z \geq 0.$$

One of the faces of the pyramid is the plane  $\Sigma : x + y + z = 1$ .

- a) (4 points) Find the distance of the opportunity camera point  $P = (3, 4, 10)$  to  $\Sigma$ .
- b) (3 points) Parametrize the line  $L$  through  $P$  which is perpendicular to  $\Sigma$ .
- c) (3 points) At which point does  $L$  intersect  $\Sigma$ ?



**Solution:**

- a) The normal vector to the plane is  $[1, 1, 1]^T$ . Use the distance formula  $d = |\vec{PQ} \cdot \vec{n}|/|\vec{n}| = 16/\sqrt{3}$ .
- b) The point  $(3, 4, 10)$  is on the curve. The vector  $\vec{n} = [1, 1, 1]^T$  is in the curve. The parametrization is  $\vec{r}(t) = [3 + t, 4 + t, 10 + t]^T$ .
- c) Having  $x = 3 + t, y = 4 + t, z = 10 + t$  and plugging it into the equation  $x + y + z = 1$  gives  $t = -16/3$  and the point  $(-7/3, -4/3, 14/3)$ .

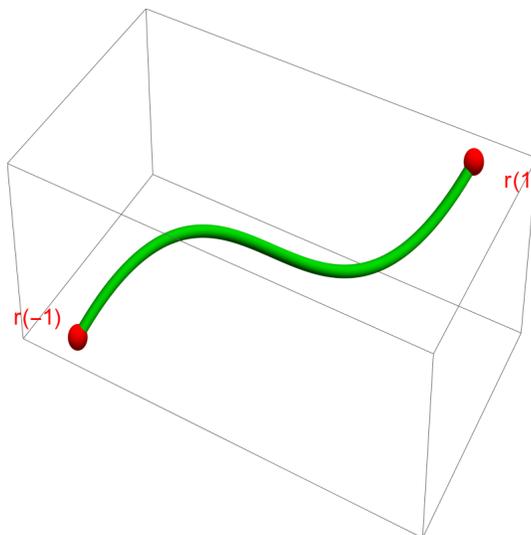
Problem 5) (10 points)

- a) (6 points) Find the arc length of the curve

$$\vec{r}(t) = \left[ t, \frac{2t^3}{3}, \frac{2t^5}{5} \right]^T$$

from  $-1$  to  $1$ .

- b) (4 points) What is the integral  $\int_{-1}^1 \vec{r}''(t) dt$ ?



**Solution:**

- a)  $\int_{-1}^1 \sqrt{1^2 + 4t^4 + 4t^8} dt = 14/5$ . (Take twice the integral from 0 to 1.)
- b) We have  $\vec{r}'(t) = [1, 2t^2, 2t^4]^T$ . The integral is the difference between the end and initial velocity is  $\vec{r}'(1) - \vec{r}'(-1) = [0, 0, 0]^T$ .

Problem 6) (10 points)

- a) (4 points) In each of the four cases, please provide an answer in form of an equation. For example, if the question is : "Example of a quadric which contains a plane but which is not a plane", a correct answer would be  $x^2 = 1$ . An other correct answer is  $x^2 = y^2$ . Give only one answer in each of the four cases. Remember that a **quadric** is an equation in  $x, y, z$  with quadratic, linear and constant terms only, which contains at least one quadratic term. With

"lines" we include "line segments".

example of a quadric which is a	Your equation
paraboloid which is not an elliptic paraboloid	
ellipsoid which is not a sphere	
quadric containing two nonparallel lines but contains no plane	
disconnected quadric containing no plane	

b) (6 points) All vectors  $\vec{v}, \vec{w}, \vec{r}', \vec{r}'', \vec{T}, \vec{T}'$  are assumed to be nonzero, The vectors  $\vec{r}, \vec{T}$  and their derivatives are evaluated at the fixed time  $t = 0$ . Check a box, if the statement above is always true.

first vector $\vec{a}$	second vector $\vec{b}$	$\vec{a} \times \vec{b} = \vec{0}$	$\vec{a} \cdot \vec{b} = 0$
$\vec{T}$	$\vec{T}'$		
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$		
$\vec{v} \times \vec{w}$	$\vec{v} + \vec{w}$		
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$		
$(\vec{v} - \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$		
$\vec{r}'$	$\vec{r}''$		

**Solution:**

- a)  $z = x^2 - y^2$ , a hyperbolic paraboloid.
- b)  $2x^2 + 3y^2 + z^2 = 1$  an ellipsoid
- c)  $z = x^2 - y^2$ , the hyperbolic paraboloid. Also the one sheeted hyperboloid would work.
- d)  $x^2 - y^2 + z^2 = -1$ , a one sheeted hyperboloid.

first vector $\vec{a}$	second vector $\vec{b}$	$\vec{a} \times \vec{b} = \vec{0}$	$\vec{a} \cdot \vec{b} = 0$
$\vec{T}$	$\vec{T}'$		X
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v} + \vec{w}$		
$\vec{v} \times \vec{w}$	$\vec{v}$		X
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$	X	
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	X	
$\vec{r}'$	$\vec{r}''$		

Problem 7) (10 points)

On July 14, 2015, a space probe will pass Pluto and offer a first look at the distant dwarf planet and its moon Charon. Let

$$\vec{r}(t) = [1 + 100t, t^2, t^3]^T$$

be the path of the probe. and let  $A = \vec{r}(0), B = \vec{r}(1), C = \vec{r}(-1)$  be three points on the path and let  $\Sigma$  be the plane through  $A, B, C$ .

- a) (4 points) Find a normal vector to  $\Sigma$ .
- b) (4 points) Find the unit tangent vector  $\vec{v} = \vec{T}'(0)$ .
- c) (2 points) Is this vector  $\vec{v}$  parallel to  $\Sigma$ ?



"New Horizons" is approaching its destination "Pluto". Artist rendering, NASA/JHUAPL/SwRI.T

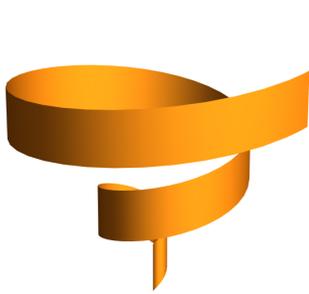
**Solution:**

- a)  $\vec{AB} \times \vec{AC} = [-2, 0, 200]^T$ .
- b)  $\vec{r}'(t) = [100, 2t, 3t^2]^T$ .  $\vec{r}'(0) = [100, 0, 0]^T$ . and  $\vec{T}'(0) = [1, 0, 0]^T$ .
- c) The plane has the normal vector  $[2, 0, 200]^T$ . The unit tangent vector is not perpendicular to this vector. The answer is "no".

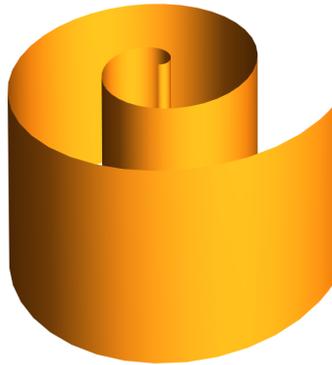
Problem 8) (10 points)

We look at six parametrized surfaces for which in each case the parameters are chosen to be in  $0 \leq t \leq 2\pi$  and  $0 \leq s \leq 3$ . In the left two columns of the following table, check the ones which apply. In the right column, enter A) - F). There is an exact match.

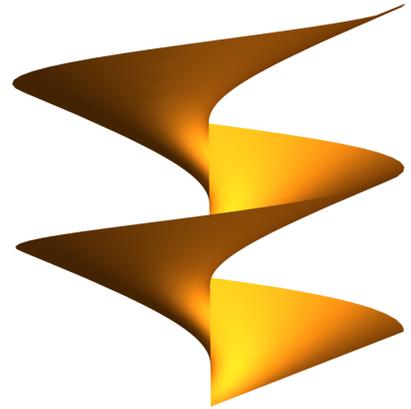
Parametrization $\vec{r}(t, s) =$	Some grid curves are lines	Some grid curves are circles	A-F
$[\cos(t), \sin(t), s + t]^T$			
$[t \cos(t), t \sin(t), s + t]^T$			
$[s \cos(t), s \sin(t), s + t]^T$			
$[t \cos(t), t \sin(t), s]^T$			
$[\cos(t), \sin(t), s]^T$			
$[s \cos(t), s \sin(t), s]^T$			



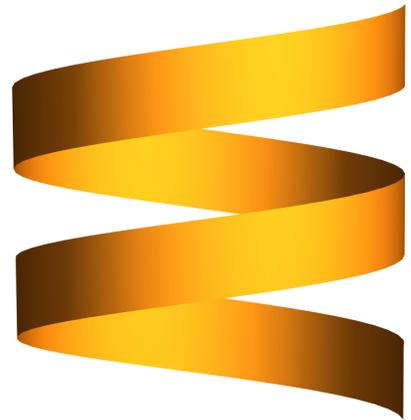
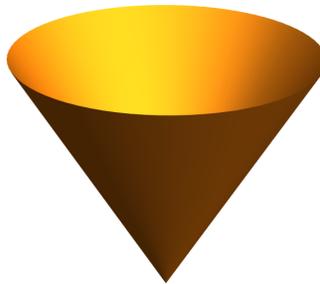
A



B



C



D

E

F

**Solution:**

F,A,C,B,D,E.

All surfaces have grid curves which are lines.

All the lower three surfaces have circles as grid curves.

Problem 9) (10 points)
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When the Philae lander left the Rosetta space craft and bounced onto the comet Chury, it felt an acceleration

$$\vec{r}''(t) = [t, t^2 - e^t, \frac{-1}{50}]^T$$

Assume  $\vec{r}(0) = [0, 0, 100]^T$  and  $\vec{r}'(0) = [1, 0, 0]^T$ . Find the curve  $\vec{r}(t)$  and the landing spot on  $z = 0$ .



The Phila lander still has again a (still spotty) connection to Rosetta. On 5th of July, 2015, a “Daily mail article” mentioned: ”Distinct features of the comet 67P/Churyumov-Gerasimenko, such as its organic-rich black crust, are best explained by the presence of living organisms beneath an icy surface”.

**Solution:**

Integrate twice and fix the initial velocities and positions.

$$\vec{r}'(t) = [t^2/2 + t, t^4/12 - e^t + t, -t^2/100]^T + [0, 1, 100]^T.$$

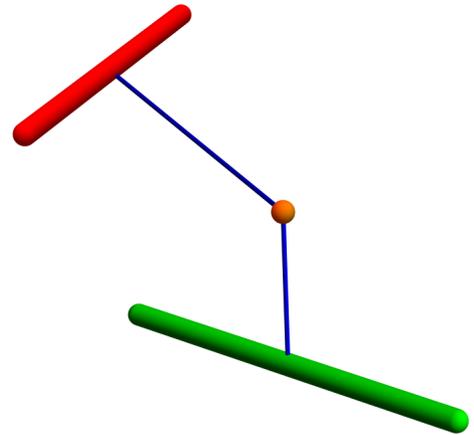
$\vec{r}(t) = [t^3/6 + t, t^4/12 - e^t + t + 1, 100 - t^2/100]^T$ . The last coordinate is zero if  $t = 100$ . This gives the landing spot  $\vec{r}(100) = [100^3/6 + 100, 100^5/12 - e^{100} + 101, 0]^T$ .

Problem 10) (10 points)
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Given a line  $\vec{r}(t) = [1 + t, t, t]^T$  and a line  $\vec{s}(t) = [1 - t, 1 + t, 1 - t]^T$ .

a) (6 points) Find the sum of the distances of the point  $(0, 0, 0)$  to the two lines.

b) (4 points) Find the distance between the two lines.



### Solution:

This is a routine distance problem, but three problems.

a) The first distance is  $|[1, 0, 0]^T \times [1, 1, 1]^T|/\sqrt{3} = \sqrt{2}/\sqrt{3}$ . The second distance is  $|[1, 1, 1]^T \times [-1, 1, -1]^T|/\sqrt{3} = \sqrt{8}/\sqrt{3} \sqrt{6}$ . In both cases, we have computed the area of a parallelogram and divided by the base length.

b) Also here, we think about geometry and get the volume of the parallel epiped spanned by  $[1, 1, 1]^T, [-1, 1, -1]^T$  and  $[0, 1, 1]^T$  (which is 2) divided by the area of the parallelogram spanned by  $[1, 1, 1]^T, [-1, 1, -1]^T$  which is the length of their cross product (which is  $\sqrt{4 + 4} = \sqrt{8}$ ). The answer is  $2/\sqrt{8} = 1/\sqrt{2}$