8/6/2020 FINAL EXAM — Maths 21a, O.Knill, Summer 2020

**"I affirm my awareness of the standards of the Harvard College Honor Code."**

Name:

Please email the PDF as an email attachment to knill@math.harvard.edu. The file needs to have your name capitalized like OliverKnill.pdf. Use **your personal handwriting**, no typing. No books, calculators, computers, or other electronic aids are allowed. (You can use a tablet to write). You can consult with a double sided page of your own handwritten notes, when writing the exam. The exam needs to arrive on Friday, August 7 at 10 AM. Write clearly and always give details of your computations. If you use separate paper, sign it with the honor code statement, use a page for each problem and copy the structure of the **check boxes**. All your final answers need to be in the boxes.

Problem 1) (20 points) No justifications are needed.

1) T F The vector $[0, 2, 1]/5$ is a direction vector.

2) T F Let $P$ be a point and $U, V$ be two spheres. The distances satisfy the inequality $d(P, U) + d(P, V) \geq d(U, V)$.

3) T F At a local minimum $(x_0, y_0)$ of $f(x, y)$ we always have $f_{xx}(x_0, y_0) > 0$.

4) T F For the unit sphere $S$ and $\vec{F} = [x^2yz, xy, xz]$, the flux $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ is zero.

5) T F The curve $\vec{r}(t) = [t, t, t^2]$ intersects the plane $x+y+2y = 5$ in a right angle.

6) T F The surface $(x - 5)^2 - (y + 1)^2 + z^2 = -1$ is a two-sheeted hyperboloid.

7) T F If $\vec{F}$ is a vector field with $|\vec{F}(x, y, z)| = 1$ for all points, and $|S|$ is the surface area of a surface $S$, then $\iint_S \vec{F} \cdot d\vec{S} \leq |S|$.

8) T F For any unit vector $\vec{v}$ we have $|\vec{v} \cdot [1, 0, 0]| \leq 1$.

9) T F If the acceleration $\vec{r}\,''(t) = 0$ at all $t$, then $\vec{r}(t)$ moves on a straight line.

10) T F For a field $\vec{F}$ and curve $\vec{r}$, we have $\frac{d}{dt}\vec{F}(\vec{r}(t)) = \text{curl}(\vec{F})(\vec{r}(t)) \cdot (\vec{r}(t)) \times \vec{r}\,'(t))$.

11) T F If $\vec{F}$ and $\vec{G}$ are both gradient fields, then $\vec{F} + \vec{G}$ is a gradient field.

12) T F $\vec{r}(s, t) = [\cos(t^3)\sin(s^3), \sin(t^3)\sin(s^3), \cos(s^3)]$ parametrizes a sphere.

13) T F The flux of $\vec{F} = [0, 7y, -4z]$ through the sphere $x^2 + y^2 + z^2 = 1$ oriented outwards is equal to $4\pi$.

14) T F The set of points satisfying $\phi^2 = 1$ in spherical coordinates is a double cone.

15) T F There exists a vector field $\vec{F}$ and a scalar function $g(x, y, z)$ such that $\text{curl}(\vec{F}) = \text{grad}(g)$.

16) T F The solid defined by $x^2 + y^2 + z^2 \leq 9, z \leq \sqrt{x^2 + y^2}$ has volume $\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2 \sin(\phi)\, d\rho d\phi d\theta$.

17) T F For any vector field $\vec{F} = [P, Q, R]$, we have $|\text{div}(\vec{F})| \leq |\text{curl}(\vec{F})|$.

18) T F $\text{grad}(\text{div}(\vec{F})) = \text{curl}(\text{curl}(\vec{F}))$.

19) T F George Green from Green's theorem was an English aristocrat.

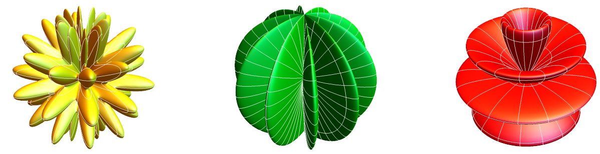
20) T F If $\vec{N}(t)$ is the normal vector in the TNB-frame to a curve $\vec{r}(t)$, then $|\vec{N}\,'(t)|/|\vec{r}\,'(t)|$ is the curvature.

| Problem 2) (10 points) No justifications are necessary. |
|---|

a) (2 points) Match the surfaces given in spherical coordinates. There is an exact match.

| Surface | A-C |
|---|---|
| $\rho = \sin^2(5\phi)$ | |
| $\rho = \sin^2(5\theta)$ | |
| $\rho = \sin^2(5\theta) + \sin^2(5\phi)$ | |

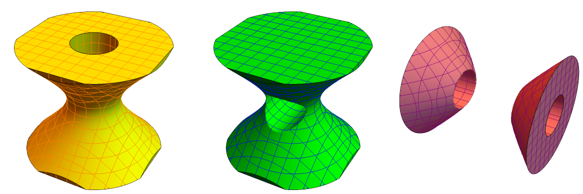
A B C



b) (2 points) Match the solids. There is an exact match.

| Solid | A-C |
|---|---|
| $x^2 + y^2 - z^2 < 2, x^2 + z^2 > 1$ | |
| $x^2 + y^2 - z^2 < 2, x^2 + y^2 > 1$ | |
| $x^2 - y^2 - z^2 > 1, y^2 + z^2 > 1$ | |

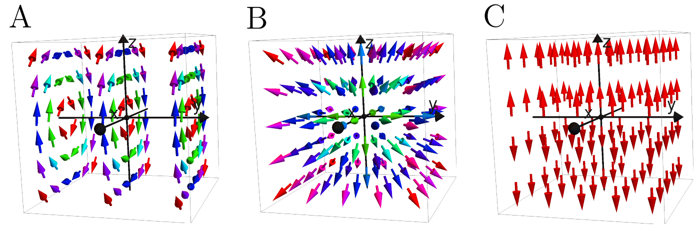
A B C



c) (2 points) The figures display vector fields. There is an exact match.

| Field | A-C |
|---|---|
| $\vec{F} = [0, 0, z]$ | |
| $\vec{F} = [x, y, z]$ | |
| $\vec{F} = [-z, 0, x]$ | |

A B C



d) (2 points) Recognize partial differential equations! You saw Klein-Gordon in the homework.

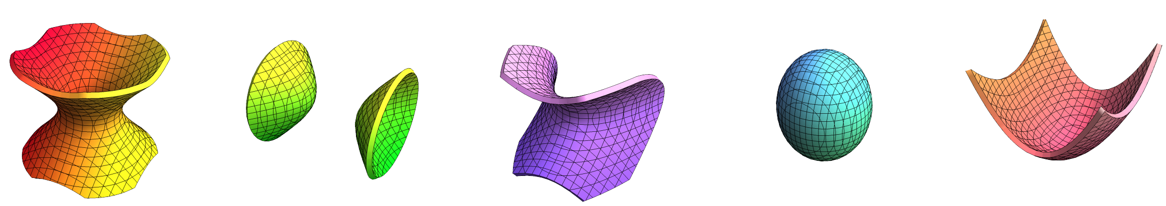
| Equation | A-F |
|---|---|
| Transport | |
| Burger | |
| Laplace | |
| Klein-Gordon | |

| | PDE |
|---|---|
| A | $g_T = g_X$ |
| B | $g_{TT} = -g_{XX}$ |
| C | $g_T^2 - 1 = g_X^2$ |

| | PDE |
|---|---|
| D | $g_T = g_X g_{XX}$ |
| E | $g_{XX} - g_{TT} = g$ |
| F | $g_{XX} + g g_X = g_T$ |

e) (2 points) Some quadrics

| | Enter a letter from A-E each |
|---|---|
| Pick the one sheeted hyperboloid | |
| Pick the ellipsoid | |
| Pick the hyperbolic paraboloid | |



A B C D E

Problem 3) (10 points) No justifications necessary

Fill in all the numbers 1-10 exactly once. While there could be multiple solutions to an individual question, the match is perfect.

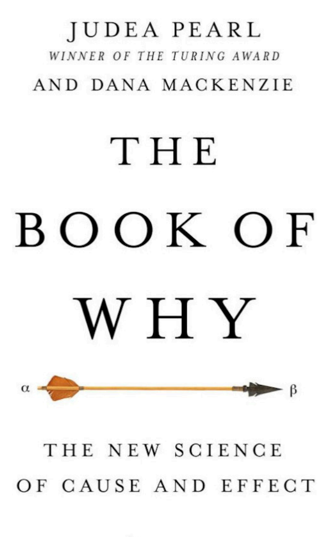
| Why? | Because (enter 1-10) |
|---|---|
| Why are gradients of f and g parallel when maximizing f under g=0? | |
| Why is $\text{div}(\text{curl}(F)) = 0$? | |
| Why is the gradient $\nabla f(x, y)$ zero at maxima or minima of $f$? | |
| Why can one change the order of integration for rectangles? | |
| Why is the line integral of a gradient field along a circle 0? | |
| Why is the flux of the curl through a sphere zero? | |
| Why is volume of a solid determined by the boundary? | |
| Why is the gradient of a function perpendicular to the level curve? | |
| Why is $\|\vec{v} \cdot \vec{w}\|/(\|\vec{v}\|\|\vec{w}\|) \leq 1$ for any $\vec{v}, \vec{w}$? | |
| Why is the area of a region determined by its boundary? | |

1 Stokes theorem, 2 Chain rule, 3 Clairaut, 4 Cauchy-Schwarz, 5 Green, 6 Fubini, 7 Lagrange, 8 Fermat, 9 Divergence theorem, 10 Fundamental theorem of line integrals.

The question Why is important as the literature displayed to the right indicates.

Curiosity is the motor of science.

Asking good questions is pivotal for understanding.

Problem 4) (10 points)

We explore our surroundings with a **small drone** of less than a quarter kilograms. It climbs with an acceleration

$$\vec{r}\,''(t) = [0, 0, 2 + \sin(t)]$$

with initial velocity $\vec{r}\ '(0) = [1, 0, 0]$ and initial position $\vec{r}(0) = [3, 4, 10]$.

a) (6 points) Where is the drone at time $t = 2\pi$?

b) (4 points) What is the **curvature** $\kappa(\vec{r}(0))$ of the curve $\vec{r}(t)$ at $t = 0$?

Problem 5) (10 points)

a) (2 points) Find the **tangent plane** to the level surface

$$g(x,y,z) = 2x^2 + y^3 - z = 0$$

at the point $(2,1,9)$.

b) (2 points) Remember that the level curve $f(x,y) = 2x^2 + y^3 = 9$ is a curve in the plane. Find the **tangent line** to the point $(2,1)$.

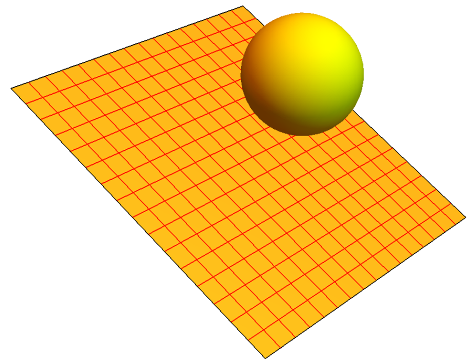
c) (2 points) Estimate the value of $2 \cdot 2.001^2 + 0.99^3$ using **linearization**.

d) (2 points) What is the **directional derivative** $D_{[0,1]}f(2,1)$?

e) (2 points) Check the **implicit differentiation formula** $f_x(2,1) = -g_x(2,1,9)/g_z(2,1,9)$.

Problem 6) (10 points)

a) (4 points) Find the **distance** of the sphere $x^2 + y^2 + z^2 = 1$ to the plane containing the points $A = (4, 0, 0), B = (0, 6, 0), C = (0, 0, 2)$.

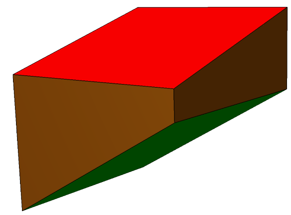


b) (3 points) What is the **implicit equation** $ax + by + cz = d$ of the plane containing $A, B, C$?

c) (3 points) What is the **area** of the triangle $ABC$?

Problem 7) (10 points)

a) (5 points) A **house** has the **ground** $z = 2y - 2$ and **roof** $z = 2 - x$ and is located above the square $0 \leq x \leq 1, 0 \leq y \leq 1$. Find the volume of the house using a **triple integral**. We want to see the triple integral!



b) (5 points) Integrate the **charge density function** $f(x, y, z) = z^4$ over the solid $E$ given by

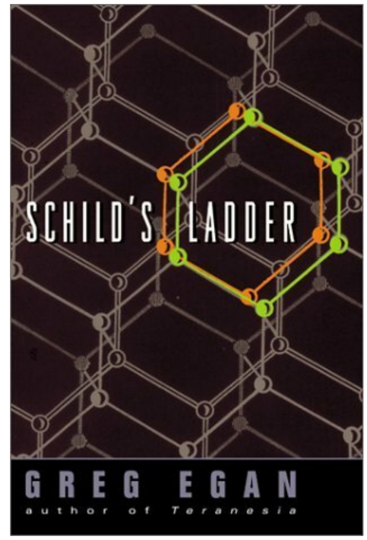
$$x^2 + y^2 + z^2 \leq 9, z > 0 \ .$$

Problem 8) (10 points)

a) (8 points) While study the **novo vacuum**, we are led to the problem to find the extrema of the function

$$f(x,y) = x^2 - \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} + y^2$$

and classify them using the **second derivative test**. The humanoid physicist Cass (a specialist in quantum graph theory) has already figured out that all critical points have integer coordinates and wants you to know this.




b) (2 points) Answer whether there is a global maximum or global minimum for $f$.

We are motivated by the novel "Schild's ladder" of Greg Egan tells the story of some humanoid physicists who need to escape and at the same time under an expanding **bubble of novo vacuum**, which is more stable than the ordinary vacuum. Mathematically, it is possible that our vacuum is only a local minimum of an energy functional and that there is a global minimum nearby. By the way, Schild's ladder is a tough book to read. It belongs to the category of hard science fiction.

Problem 9) (10 points)

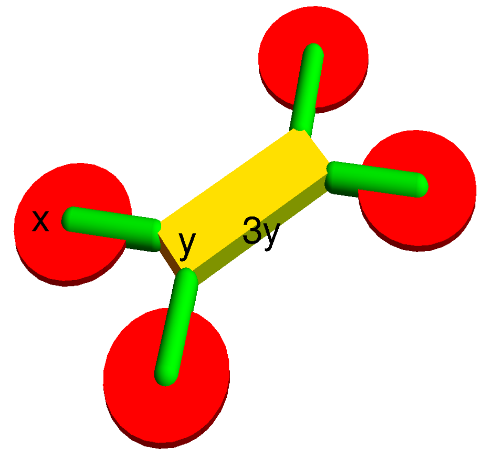
We want to minimize the weight

$$f(x,y) = 4x^2 + 9y^2$$

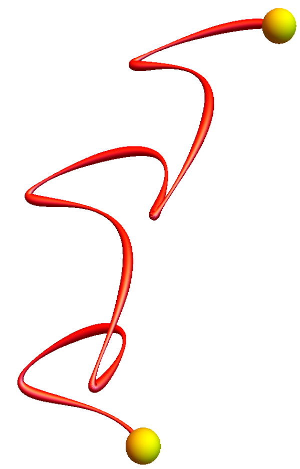
covered by four propellers of weight $x^2$ each and body of a **micro drone** with weight $9y^2$ while working with the constaint that the skeleton length

$$g(x,y) = 4x + 8y = 100$$

is constant. You need to solve the problem with the method we have learned in this course, even if you can solve the problem differently. The smallest micro drones one can buy have the size of **dragon flies**. Robotics labs work on drones and robo bees having the size of an actual fly.



Problem 10) (10 points)



What is the line integral of

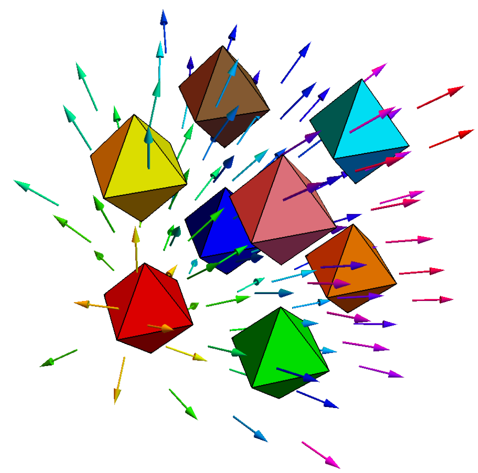
$$\vec{F}(x,y,z) = [3x^3 + yz, 3y^3 + xz, 3z^3 + xy]$$

along the curve

$$\vec{r}(t) = [t^2 + 3t + \sin(\sin(\pi 15t)), t^3, t^2 + \sin(\sin(\pi 15t))] ,$$

where $0 \leq t \leq 1$?

Problem 11) (10 points)



As part of an art project **"octogrid of octahedra"**, we radiate a laser field $\vec{F}$ through a solid composed of **8 gems**, each being an octahedron of length 1. Each single octahedron is known to have volume $\sqrt{2}/3$ of course (base area time height $\sqrt{2}$ divided by 3). The radiation field is

$$\vec{F} = [5\sqrt{2}x + \sin(yz), 4\sqrt{2}y + \cos(xz), 3\sqrt{2}z + \sin(xy)] .$$

What is the total flux of $\iint_S \vec{F} \cdot d\vec{S}$ through the boundary surface $S$ of **the union of all these** 8 **gems** if the boundary is oriented outwards on each gem?

Problem 12) (10 points)

A **ribbon** $S$ is parametrized as

$$\vec{r}(t,s) = [(3+\cos(s))\cos(t), (3+\cos(s))\sin(t), \sin(s)] \ ,$$

with two parameters

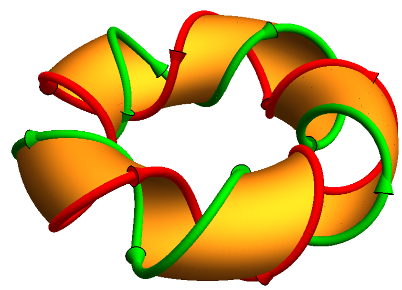
$$0 \le t \le 2\pi, 5t - \pi \le s \le 5t \ .$$

It is bounded by the curves

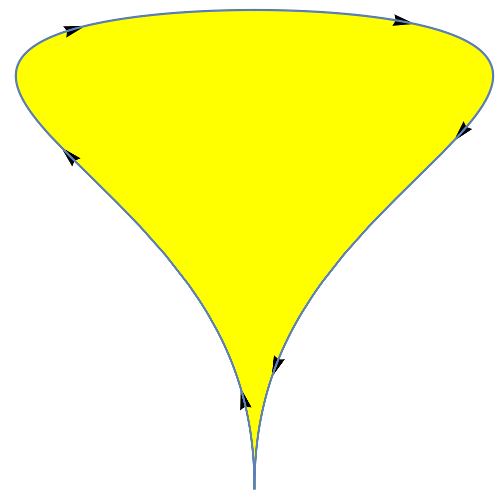
$$\vec{r}_1(t) = [\cos(t)(3+\cos(5t)), \sin(t)(3+\cos(5t)), \sin(5t)]$$

$$\vec{r}_2(t) = [\cos(t)(3-\cos(5t)), \sin(t)(3-\cos(5t)), -\sin(5t)] \ ,$$

where $t$ goes from 0 to $2\pi$. The orientation of these curves (counter clockwise when looking from above) is indicated in the picture. In each case, it might or might not be compatible with the orientation of the outwards oriented ribbon $S$. Compute the flux of the curl of $\vec{F}(x,y,z) = [0, 0, \sqrt{x^2+y^2} - 3]$ through $S$.



Problem 13) (10 points)



While eating breakfast we doodle around with some honey spread on our buttered bread. Find the area of the happily created **honey region** enclosed by

$$\vec{r}(t) = [\frac{\sin^2(\pi t)}{t}, 4 - 4t^2]$$

for $-1 \leq t \leq 1$. The curve has been traced in the picture.