

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 1: Geometry and Distance

### LECTURE

**1.1.** Points on the **real line**  $\mathbb{R}$  can be fixed by a single coordinate  $x$ . The zero  $0$  divides the **positive axis** from the **negative axis**. A point  $P$  in the **plane**  $\mathbb{R}^2$  is determined by two **coordinates**. We write  $P = (x, y)$ . In space  $\mathbb{R}^3$  finally, we require three coordinates  $P = (x, y, z)$ , where  $z$  usually is thought of as height, the distance from the  $xy$ -plane. The signs define four **quadrants** in  $\mathbb{R}^2$  or eight **octants** in  $\mathbb{R}^3$ . These regions all intersect at the **origin**  $O = (0, 0)$  or  $O = (0, 0, 0)$  and are bound by **coordinate axes**  $\{y = 0\}$  and  $\{x = 0\}$  or **coordinate planes**  $\{x = 0\}, \{y = 0\}, \{z = 0\}$ .

**1.2.** In  $\mathbb{R}^2$  we usually orient the  $x$ -axis to point “east” and the  $y$ -axis to be “north”. In  $\mathbb{R}^3$ , the most common coordinate system is to see the  $xy$ -plane as the “ground” and imagine the  $z$ -coordinate axes pointing “up”. In computer graphics or photography, the  $xy$ -plane is the **retina** or film plate and the  $z$ -coordinate measures the distance towards the viewer. In this **photographic coordinate** system, your eyes and chin define three points in the plane  $z = 0$  and the nose points into the positive  $z$  direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates  $(1, 0, 0)$  for the right eye,  $(-1, 0, 0)$  for the left eye, then the tip of your nose might have the coordinates  $(0, -1, 1)$ . If this is too abstract, draw the situation.

**1.3.** The **Euclidean distance** between two points  $P = (x, y, z)$  and  $Q = (a, b, c)$  in space is defined as

**Definition:**  $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$ .

This is a **definition** and not a result. It is motivated by the **theorem of Pythagoras**, but we will **prove** the later result in a moment. This distance is defined in any dimension. In the plane for example the distance of the point  $(x, y)$  to  $(a, b)$  is  $\sqrt{(x - a)^2 + (y - b)^2}$ . If we work in  $\mathbb{R}^2$ , we do not think of it as part of  $\mathbb{R}^3$ . Coordinates work in arbitrary dimensions. A collection of  $n$  data points defines a vector in  $\mathbb{R}^n$ . Working in Euclidean space  $\mathbb{R}^n$  makes sense from a **data scientist point of view**. One can define the Euclidean distance between  $x = (x_1, \dots, x_n)$  and  $a = (a_1, \dots, a_n)$  as  $d(x, a)^2 = \sum_{k=1}^n (x_k - a_k)^2$ . Having the sum of the squares appears in statistics in **least square problems**.

**1.4. Points, curves, surfaces and solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a **line**, an example of a surface is a **plane**, an example of a solid is the **ball**, the interior of a **sphere**.

**Definition:** A **circle** of radius  $r \geq 0$  centered at  $P = (a, b)$  is the collection of points in  $\mathbb{R}^2$  which have distance  $r$  from  $P$ . A **sphere** of radius  $\rho$  centered at  $P = (a, b, c)$  is the collection of points in  $\mathbb{R}^3$  which have distance  $\rho \geq 0$  from  $P$ . The equation of a sphere is  $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$ .

**1.5.** When **completing the square** of an equation  $x^2 + bx + c = 0$ , we add  $(b/2)^2 - c$  on both sides of the equation in order to get  $(x + b/2)^2 = (b/2)^2 - c$ . Solving for  $x$  gives  $x = -b/2 \pm \sqrt{(b/2)^2 - c}$ . This is the **quadratic equation**. Know this equation. You don't want to waste your creative power having to re-derive this again and again.

#### EXAMPLES

**1.6.**  $P = (-2, -3)$  is in the third quadrant of the plane and  $P = (1, 2, 3)$  is in the positive octant of space. The point  $(0, 0, -8)$  is located on the negative  $z$  axis. The point  $P = (1, 2, -3)$  is below the  $xy$ -plane. Can you spot the point  $Q$  on the  $xy$ -plane which is closest to  $P$ ?

**1.7. Problem:** Find the distance midpoint  $M$  of  $P = (1, 2, 5)$  and  $Q = (-3, 4, 7)$  and verify that  $d(P, M) + d(Q, M) = d(P, Q)$ . **Answer:** The distance is  $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$ . The distance  $d(P, M)$  is  $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ . The distance  $d(Q, M)$  is  $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ . Indeed  $d(P, M) + d(M, Q) = d(P, Q)$ .

**1.8.** The equation  $x^2 + 5x + y^2 - 2y + z^2 = -1$  is after a **completion of the square**  $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$  or  $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$ . We see a sphere **center**  $(5/2, 1, 0)$  and **radius**  $5/2$ .

**1.9.** The distance  $d(P, Q) = |x - a| + |y - b|$  in the plane  $\mathbb{R}^2$  is called the **taxi metric** or **Manhattan distance**. **Problem:** draw a circle of radius 2. More challenging is to draw an ellipse: the set of points whose sum of the distances from  $(-2, 0)$  and  $(2, 0)$  is equal to 6. You can do that with a neat geometric construction.

**1.10.** Draw the unit circle of the **quartic distance**  $d(x, y) = (x - a)^4 + (y - b)^4$ . More generally, for any  $p > 1$ , we get a distance  $d(x, y) = (x - a)^p + (y - b)^p$ . For  $p = 1$ , it is the **taxi metric**, for  $t = 2$  it is the **Euclidean metric**, for  $t \rightarrow \infty$  it goes to the distance  $\max(|x - a|, |y - b|)$  which is the  $l^\infty$  metric. **Problem:** is  $d(P, Q) = \sqrt{|x - a|} + \sqrt{|y - b|}$  a distance? **Answer:** no, while it satisfies  $d(P, Q) = d(Q, P)$  and is zero if and only if  $P = Q$ , it does not satisfy the **triangle inequality**  $d(A, B) + d(B, C) \geq d(A, C)$ . We call a space  $(X, d)$  for which  $d$  is a distance formula satisfying  $d(P, Q) = d(Q, P)$ ,  $d(P, Q) = 0 \Leftrightarrow P = Q$  and  $d(A, B) + d(B, C) \geq d(A, C)$  a **metric space**.

**1.11. Problem:** Find an algebraic expression for the set of all points for which the sum of the distances to  $A = (1, 0)$  and  $B = (-1, 0)$  is equal to 3. **Answer:** Square the equation  $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 3$ , separate the remaining single square root on one side and square again. Simplification gives  $20x^2 + 36y^2 = 45$  which is equivalent to  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $a, b$  can be computed as follows: because  $P = (a, 0)$  satisfies this equation,  $d(P, A) + d(P, B) = (a-1) + (a+1) = 3$  so that  $a = 3/2$ . Similarly, the point  $Q = (0, b)$  satisfying it gives  $d(Q, A) + d(Q, B) = 2\sqrt{b^2 + 1} = 3$  or  $b = \sqrt{5}/2$ .

**1.12.** In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and-compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. But ideas do not grow in a vacuum; Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. One of the first to explore higher dimensional Euclidean space was Ludwig Schläfli.<sup>1</sup>

**1.13.** The method of completion of squares is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work "*Compendium on Calculation by Completion and Reduction*" was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad.<sup>2</sup>

**1.14.** The Euclidean geometry described is only one of many geometries. One can work with more general **metric spaces**. An important class of metric spaces are studied in Riemannian geometry, where the distance between two points can become dependent on where we are. Space becomes curved. This is the frame work of general relativity. Formally, this can happen by changing the coefficients  $E, G$  of the metric  $d(P, Q)^2 = E(x-a)^2 + G(y-b)^2$ . On a sphere, where  $x = \theta \in [0, 2\pi]$  is longitude and  $y = \phi \in [0, \pi]$  is latitude, one would take  $E = \sin^2(y), G = 1$ . Two points on the arctic circle with fixed longitude have shorter distance than two points on the equator with the same fixed longitudes. It is important to think now of the surface of the sphere as a space itself, without its embedding in the ambient space. This space is curved. Our four dimensional space-time universe is curved depending on the matter distribution.

## HOMEWORK

This homework is due on Tuesday, 6/30/2020.

**Problem 1.1:** Describe in words and draw the objects in  $\mathbb{R}^3$ .

a)  $(z + 3)^2 + (y - 6)^2 = 100$ .

b)  $|x - 5| + |y - 2| + |z - 3| = 4$ .

c)  $x^2 y^2 z^2 = 0$ ,

d)  $4x + 2y + 3z = 12$ .

e)  $|(x - 1, y, z)| - |(x, y - 1, z)| = 1$ .

f)  $x^2 - z^2 = 1$ .

<sup>1</sup>An entertaining read is "Descartes secret notebook" by Amir Aczel which deals with an other discovery of Descartes.

<sup>2</sup>The book "The mathematics of Egypt, Mesopotamia, China, India and Islam, by Ed Victor Katz, page 542 contains translations of some of this work.

**Problem 1.2:** A data point  $P = (20, 6, 8)$  gives temperature, rain and wind velocity. We can visualize this in the  $(t, r, v)$  space. a) Find the distance of  $P$  to the  $t$ -axes, where  $r, v$  are zero. b) Find the center of the sphere  $t^2 + r^2 - 20r + v^2 = 21$ .

**Problem 1.3:** Verify that the radius of the inscribed circle in a 3 : 4 : 5 triangle is 1. Here is a hint: make a picture of the triangle  $ABC$  given by  $A = (0, 0), B = (4, 0), C = (0, 3)$ , introduce  $M = (1, 1)$  then get the coordinates of the points  $X, Y, Z$  then compute the distances to verify that the inscribed circle touches the triangle at  $X, Y, Z$ .

**Problem 1.4:** The figure shows rectangles of area 64 and 65 made up of matching pieces. What is going on? It is a famous and classical problem. Try first on your own!

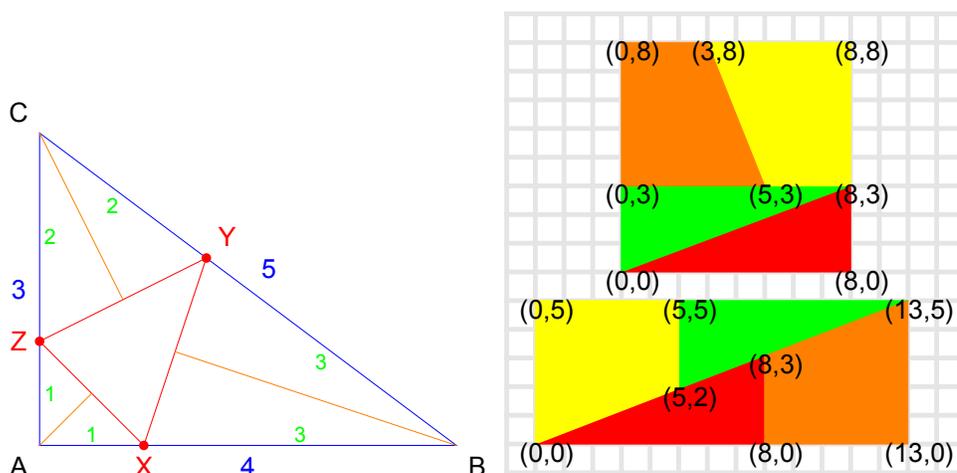


FIGURE 1. The 3-4-5 triangle and the missing square riddle.

**Problem 1.5:** You play billiard in the table  $\{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 8\}$ . a) Hit the ball at  $(3, 2)$  to reach the hole  $(4, 8)$  bouncing 3 times at the left wall and three times at the right wall and no other walls. Find the length of the shot. b) Hit from  $(3, 2)$  to reach the hole  $(4, 0)$  after hitting twice the left and twice the right wall as well as the top wall  $y = 8$  once. What is the length of the trajectory?