MULTIVARIABLE CALCULUS

MATH S-21A

Unit 2: Vectors and dot product

Lecture

2.1. Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$.

We simply write this column vector also as a row vector $[x - a, y - b, z - c]$ or in order to save space. As the vector starts at P to Q we write $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in $\vec{v} = [p, q, r]$ are called the **components** of $\vec{v}$.

2.2. Vectors can be placed **anywhere** in space. [1] Two vectors with the same components are considered **equal**. Vectors can be translated into each other if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]$ heads to the point (p, q, r). One can therefore identify **points** $P = (a, b, c)$ with **vectors** $\vec{v} = [a, b, c]$ attached to the origin. For clarity, we often draw an arrow $\vec{}$ on top of a vector variable and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to write $[2, 3, 4]$ for vectors and $(2, 3, 4)$ for points.

2.3.

> **Definition:** The **sum** of two vectors is $\vec{u}+\vec{v} = [u_1, u_2]+[v_1, v_2] = [u_1+v_1, u_2+v_2]$. The **scalar multiple** is $\lambda\vec{u} = \lambda[u_1, u_2] = [\lambda u_1, \lambda u_2]$. The **difference** $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

Commutativity, associativity, or distributivity rules for vectors are inherited directly from the corresponding rules for numbers.

2.4. The vectors $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, $\vec{k} = [0, 0, 1]$ are called **standard basis vectors**. We will avoid this notation mostly but it has historically grown as some notions like the dot and cross product have grown from **quaternions** which are points (t, x, y, z) in $\mathbb{R}^4$ usually written as $t + ix + jy + kz$.

[1] In differential geometry, a vector $\vec{v}$ attached at P is in the tangent space to a point P.

2.5.

Definition: The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P,Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is called a **direction** of $\vec{v}$. The only vector of length 0 is the 0 vector $[0,0,0]$.

2.6.

Definition: The **dot product** of two vectors $\vec{v} = [a,b,c]$ and $\vec{w} = [p,q,r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

2.7. Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, the **Dirac notation** $\langle \vec{v} | \vec{w} \rangle$ is used in quantum mechanics or the **Einstein notation** $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity is used. In statistics, it is called the **covariance** $\text{Cov}[v,w]$ of centered data points. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product $g(v,w)$ which is linear in v and w and satisfies the symmetry $g(v,w) = g(w,v)$ and $g(v,v) \geq 0$ and $g(v,v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v,w) = 2v_1w_1 + 3v_2w_2 + 5v_3w_3$.

2.8. The dot product determines distances and distances determines the dot product. **Proof:** Write $v = \vec{v}$. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v+w)\cdot(v+w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

$$v \cdot w = (|v+w|^2 - |v|^2 - |w|^2)/2 \,.$$

2.9. The **Cauchy-Schwarz inequality** is

Theorem: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. If $|w| = 0$, the statement holds as both sides are zero. Otherwise, assume $|w| = 1$ by dividing the equation by $|w|$. Now plug in $a = v \cdot w$ into the equation $0 \leq (v-aw)\cdot(v-aw)$ to get $0 \leq (v-(v\cdot w)w)\cdot(v-(v\cdot w)w) = |v|^2 + (v\cdot w)^2 - 2(v\cdot w)^2 = |v|^2 - (v\cdot w)^2$ which means $(v\cdot w)^2 \leq |v|^2$. □

2.10. Having established this, it is possible to give a definition of what an **angle** is, without referring to any geometric pictures:

Definition: The **angle** between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0,\pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Since cos maps $[0,\pi]$ in a 1:1 manner to $[-1,1]$, this is well defined.

2.11. The **Al Kashi's theorem** gives the third side length c of a triangle ABC in terms of the sides $a = d(B,C), b = d(A,C)$ and α, the angle at the vertex C

Theorem: $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$. □

2.12. The **triangle inequality** tells

Theorem: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof. $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$. □

Definition: Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$.

2.13. We can now prove the **Pythagoras theorem**:

Theorem: If $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof. $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. □

2.14.

Definition: The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.

2.15. The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Examples

2.16. For example, with $\vec{v} = [0, -1, 1]$, $\vec{w} = [1, -1, 0]$, $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]$. Its length is $1/\sqrt{2}$.

2.17. The RGB **color space** consists of triples $\vec{v} = [r, g, b]$ describing the amount of red, green and blue of a **color**. An other coordinate system is the CMY **color space** consisting of triples $\vec{v} = [c, m, y] = [1 - r, 1 - g, 1 - b]$, where c is **cyan**, m is **magenta** and y is **yellow**.

2.18. In physics, forces and fields $\vec{F}$ are described by vectors. The **velocity** of a curve $r(t) = [x(t), y(t), z(t)]$ is a vector attached to the point $r(t)$.

2.19. In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear.

Homework

This homework is due on Tuesday, 6/30/2020.

Problem 2.1: a) Find a **unit vector** parallel to $\vec{x} = \vec{u} + \vec{v} + 2\vec{w}$ if $\vec{u} = [-1, 0, 1]$ and $\vec{v} = [1, 1, 0]$ and $\vec{w} = [0, 1, 1]$.
b) Now find a unit vector perpendicular to $\vec{x}$. (there are many solutions).

Problem 2.2: An **Euler brick** is a **cuboid** with side lengths a, b, c such that all face diagonals are integers.
a) Verify that $\vec{v} = [a, b, c] = [275, 252, 240]$ is a vector which leads to an Euler brick. Halcke found the first one in 1719.
b) (*) Verify that $[a, b, c] = [u(4v^2 - w^2), v(4u^2 - w^2), 4uvw]$ leads to an Euler brick if $u^2 + v^2 = w^2$.
(Sounderson 1740) If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

Problem 2.3: **Colors** are encoded by vectors $\vec{v} =$ [red , green , blue]. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.
a) Determine the angle between the colors yellow and magenta.
b) What is the vector projection of the magenta-orange mixture $\vec{x} = (\vec{v} + \vec{w})/2$ onto green $\vec{y}$?

Problem 2.4: A rope is wound exactly 8 times around a stick of circumference 1 and length 15. How long is the rope?

Problem 2.5: a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.
b) Find the vector projection of the main diagonal $\vec{v} = [1, 1, 1]$ onto the side diagonal $\vec{w} = [1, 1, 0]$.
c) Find the maximal distance between the 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$ of a **tesseract**.

2.20. We live in a time, where **data** are increasingly important. This is good news for multi-variable calculus, as data points are usually given as points in an Euclidean space and analyzed using tools of multi-variable calculus. There are various ways how data can be stored, in a computer as lists of **bits**, in a quantum computer as a list of **qbits**, in a **relational database** as a list of tables, in a **graph data base** as a list of graphs. A sort of graph database has been designed already by the Incas in the form of **Khipu**, which are also called "talking knots". In a picture, data are color values attached in an array, a **song** is an array of amplitudes, a movie is an array of pictures, which each is an array of color vectors. It does not matter, in the end, we can store information as a point in a Euclidean space. The Harvard Khipu data base for example is a standard relational database encoding the three dimensional knots which are still available. We look here at an example, which illustrates the **story of data**:

2.21. The next table shows some data from the 2018 rankings of the top universities. The data were obtained from the website

$$https://www.topuniversities.com/university-rankings/world-university-rankings/2018$$

There are different such rankings. It is important that you are aware about how arbitrary such a "ranking" can be done. This is especially if you think about how the data were actually obtained. Which of the data can be considered objectively reproducible, which ones are more subjective?

2.22. The following concrete data provide for each institution a vector:

[Overall,Academic,Employer,Faculty/Student ratio,Citations,Int. faculty, Int. students]

MIT	100	100	100	100	99.8	100	95.5
Stanford	98.6	100	100	100	99	99.8	70.5
Harvard	98.5	100	100	99.3	99.8	92.1	75.7
Caltech	97.2	98.7	81.2	100	100	96.8	90.3
Oxford	96.8	100	100	100	83	99.6	98.8
Cambridge	95.6	100	100	100	77.2	99.4	97.9
ETH	95.3	98.2	96.2	82.4	98.7	100	98.6
Imperial	93.3	98.7	99.9	99.9	67.8	100	100
Chicago	93.2	99.6	90.7	97.4	83.6	74.2	82.5
UCL	92.9	99.3	99.2	99.2	66.2	98.7	100

2.23. Can you figure out how the first entry is computed? It is an average of the 6 entries
$\vec{x} = [Academic, Faculty - StudentRatio, Citations, International, IntStudents]$.
There is a mystery vector $\vec{v} = [a, b, c, d, e, f]$ which averages the ranking as a **dot product** of $\vec{x}$ with $\vec{v}$. Check also

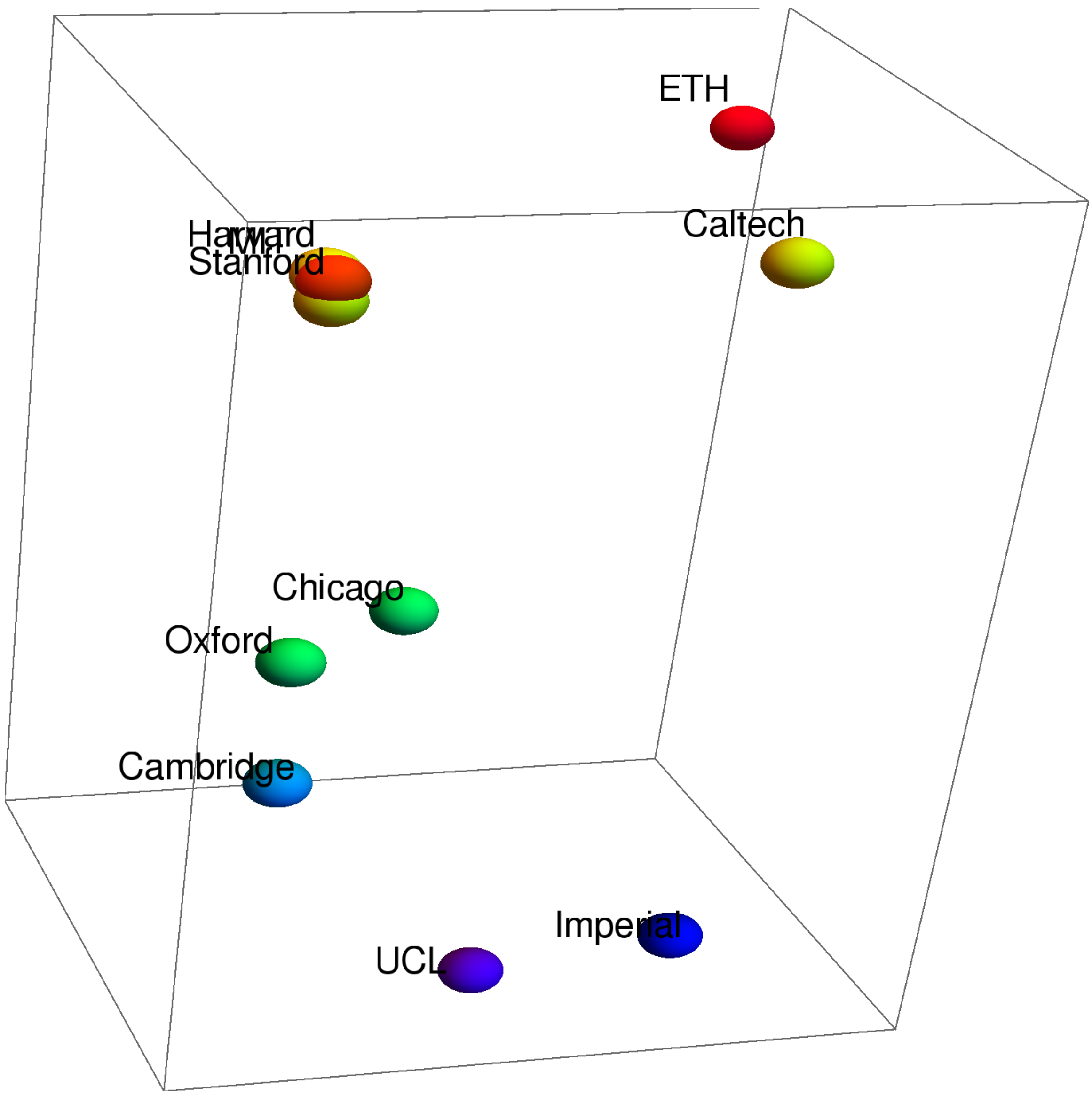
$$https://www.topuniversities.com/qs-world-university-rankings/methodology.$$

2.24. Obviously, the overall ranking depends on the key. There is a ranking key so that the university UCL is on the top for example. Can you find one? This requires to find a ranking vector $\vec{v}$ for which $\vec{x} \cdot \vec{v}$ is maximal for UCL.

2.25. How can we **visualize data**? In the next figure, we plot three relevant data points given by

$$\vec{x} = [Academic, Faculty - StudentRatio, Citations]$$

By comparing the data with the plot, can you figure out, what each of the coordinate axes is? Also this visual representation produces some kind of ranking. Being a graduate of ETH myself and being at Harvard, having been at Caltech, the picture has been turned so that these universities look particularly good ... It is important that you are aware of such manipulations. They are everywhere!



The mystery vector is $\vec{v} = [0.4, 0.1, 0.2, 0.2, 0.05, 0.05]$.