MULTIVARIABLE CALCULUS

MATH S-21A

Unit 10: Linearization

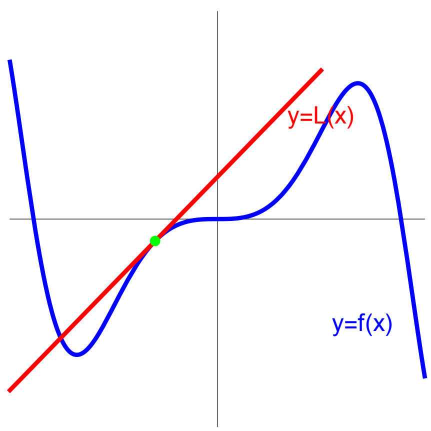
LECTURE

10.1. You know from single variable calculus already:

Definition: The **linear approximation** of $f(x)$ at a is the affine function

$$L(x) = f(a) + f'(a)(x - a) .$$

10.2. If you should have seen **Taylor series**, this is the part of the series $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k/k!$ where only the $k = 0$ and $k = 1$ term are considered. We think about the linear approximation L as a function and not as a graph because we will also look at linear approximations for functions of three variables, where we can not draw graphs.



10.3. The graph of the function L is close to the graph of f at a. What about higher dimensions?

Definition: The **linear approximation** of $f(x, y)$ at (a, b) is the affine function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) .$$

The **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) .$$

10.4. Using the **gradient**

$$\nabla f(x,y) = [f_x, f_y], \qquad \nabla f(x,y,z) = [f_x, f_y, f_z] ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .$$

10.5. How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x,b) = f(a,b) + f_x(a,b)(x-a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0,y) = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

Examples

10.6. What is the linear approximation of the function $f(x,y) = \sin(\pi xy^2)$ at the point $(1,1)$? Answer: We have $[f_x(x,y), f_y(x,y)] = [\pi y^2 \cos(\pi xy^2), 2xy\pi\cos(\pi xy^2)]$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = [\pi\cos(\pi), 2\pi\cos(\pi)] = [-\pi, -2\pi]$. The function is $L(x,y) = 0 + (-\pi)(x-1) - 2\pi(y-1)$.

10.7. Linearization can be used to estimate functions near a point. In the previous example,

$$f(1+0.01, 1+0.01) = -0.0095$$

$$L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 = -3\pi/100 = -0.00942 .$$

10.8. Here is an example in three dimensions: find the linear approximation to $f(x,y,z) = xy+yz+zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = (2,2,2)$. we have $L(x,y,z) = f(1,1,1) + (2,2,2)\cdot(x-1,y-1,z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

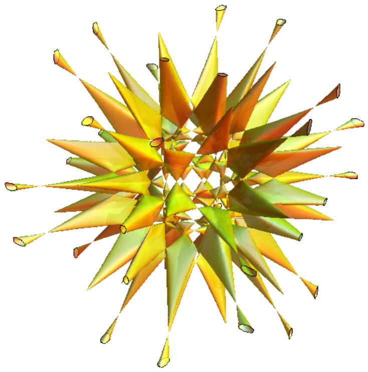
10.9. Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x\sqrt{y}z$.
Solution: take $(x_0,y_0,z_0) = (0,25,1)$, where $f(x_0,y_0,z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x\sqrt{y}z, e^x z/(2\sqrt{y}), e^x\sqrt{y})$. At the point $(x_0,y_0,z_0) = (0,25,1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5,1/10,5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5,1/10,5)\cdot(0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

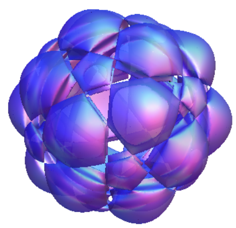
10.10. Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$.
Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $[a,b] = \nabla f(2,4) = [-g'(2), 1] = [-4, 1]$ and forming $-4x + y = d$, where $d = -4\cdot 2 + 1\cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

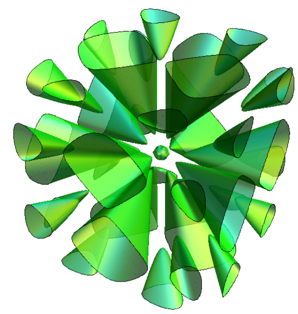
10.11. The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x, y, z) &= (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 \\ &+ 8(x^2 - t^4y^2)(-(t^4x^2) + z^2)(y^2 - t^4z^2)(x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4) , \end{aligned}$$

where $t = (\sqrt{5} + 1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5} - 1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x, y, z) = 8$. Find the tangent plane of the later surface at the point $(1, 1, 0)$.
Answer: We have $\nabla f(1, 1, 0) = [64, 64, 0]$. The surface is $x + y = d$ for some constant d. By plugging in $(1, 1, 0)$ we see that $x + y = 2$.



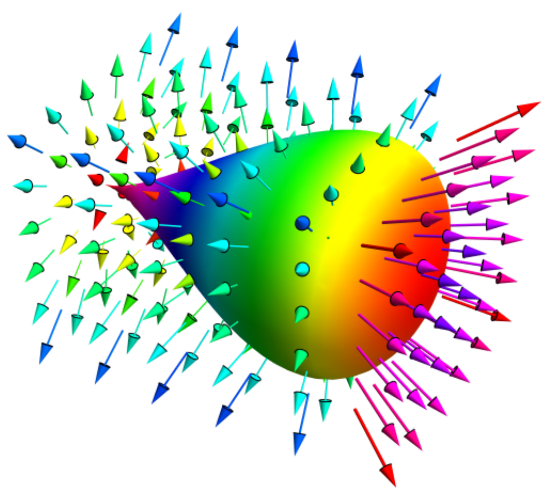




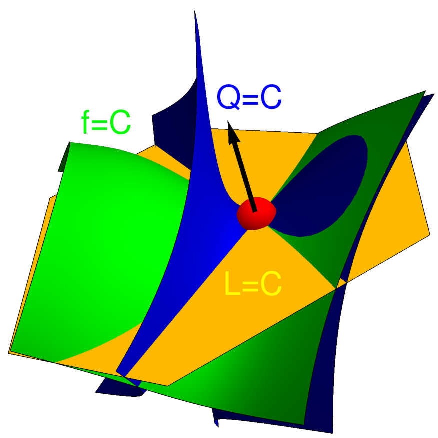
The quartic surface

$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2, 2, 2)$ of this pair shaped surface? We get $[a, b, c] = [20, 4, 4]$ and so the equation of the plane $20x+4y+4z = 56$, where we have obtained the constant to the right by plugging in the point $(x, y, z) = (2, 2, 2)$.



10.12. Linearization is just the first step for more accurate approximations. One could do **quadratic approximations** for example. In one dimension, one has $Q(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$. In two dimensions, this becomes $Q(x,y) = L(x,y) + H(a,b)[x-a, y-b] \cdot [x-a, y-b]/2$, where H is the **Hessian matrix** $H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$. We will see this matrix next week, when we maximize or minimze functions.



Homework

This homework is due on Tuesday, 7/14/2020.

Problem 10.1: Estimate $200'000'000'000'000^{1/11}$ using linear approximation of $f(x) = x^{1/11}$ near $x_0 = 20^{11}$.

Problem 10.2: Given $f(x,y) = \frac{3yx}{\pi} - \cos(x)$. Estimate $f(\pi + 0.01, \pi - 0.03)$ using linearization

Problem 10.3: Estimate $f(0.003, 0.9999)$ for $f(x,y) = \cos(\pi y) + \sin(x + \pi y)$ using linearization.

Problem 10.4: Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

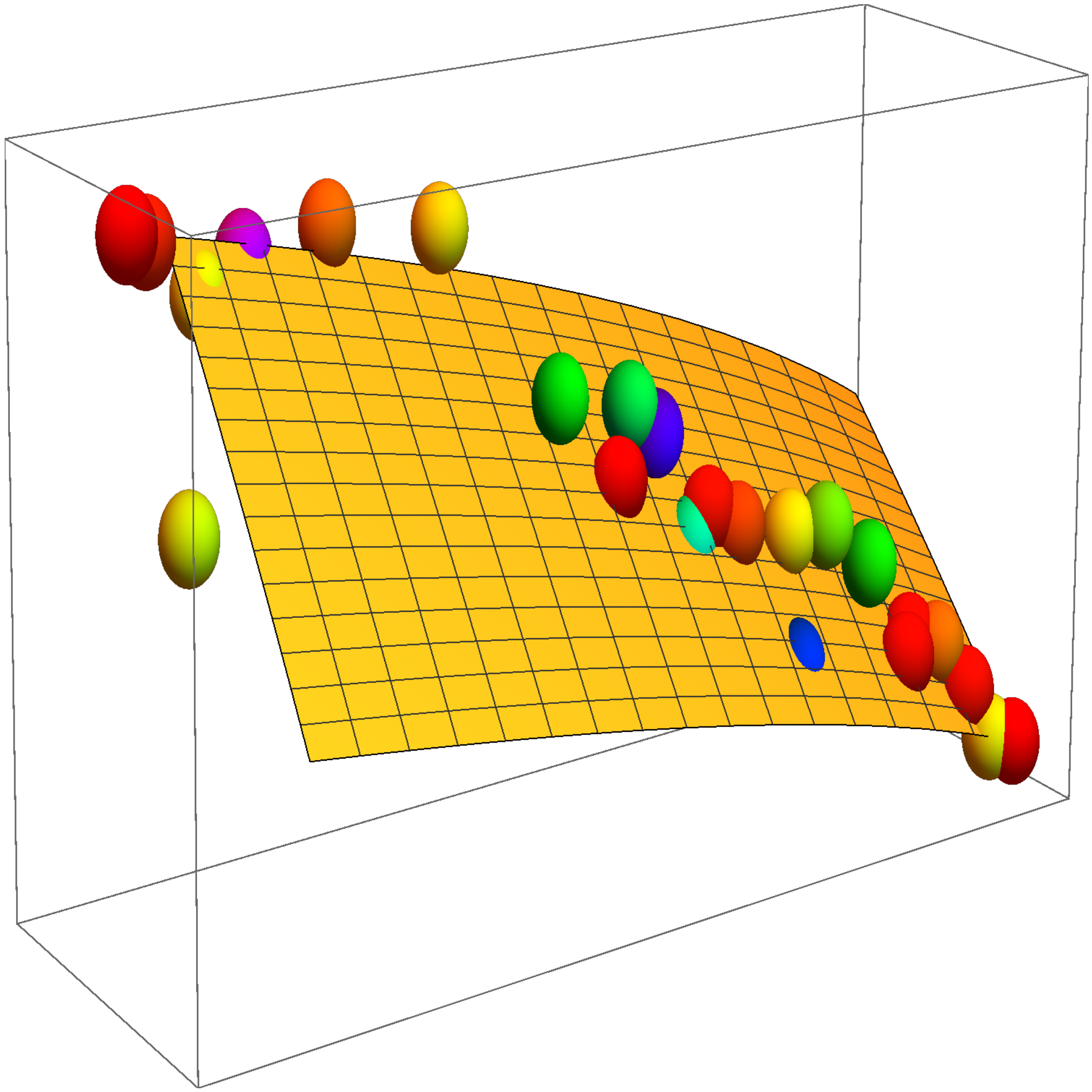
at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

Problem 10.5: Estimate $(99^3 * 101^2)$ by linearizing the function $f(x,y) = x^3y^2$ at $(100,100)$. What is the difference between $L(100,100)$ and $f(100,100)$?

DATA ILLUSTRATION COBB-DOUGLAS

10.13. The mathematician and economist **Charles W. Cobb** at Amherst college and the economist and politician **Paul H. Douglas** who was also teaching at Amherst, found in 1928 empirically a formula $F(K, L) = L^\alpha K^\beta$ which fits the **total production** F of an economic system as a function of the **capital investment** K and the **labor** L. The two authors used logarithms variables and assumed linearity to find α, β. Below are the data normalized so that the date for year 1899 has the value 100.

$Year$	K	L	P
1899	100	100	100
1900	107	105	101
1901	114	110	112
1902	122	118	122
1903	131	123	124
1904	138	116	122
1905	149	125	143
1906	163	133	152
1907	176	138	151
1908	185	121	126
1909	198	140	155
1910	208	144	159
1911	216	145	153
1912	226	152	177
1913	236	154	184
1914	244	149	169
1915	266	154	189
1916	298	182	225
1917	335	196	227
1918	366	200	223
1919	387	193	218
1920	407	193	231
1921	417	147	179
1922	431	161	240



The graph of $F(L,K) = L^{3/4}K^{1/4}$ fits pretty well that data set.

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