MULTIVARIABLE CALCULUS

MATH S-21A

Unit 13: Extrema

Lecture

13.1. In many applications we are led to the task to **maximize** or **minimize** a function f. As in single variable calculus we first search for points where the **derivative** is zero because this is needed at maxima by the **Fermat principle**. In one dimensions, like for $f(x) = 3x^5 - 5x^3$ we can use the second derivative test to classify extrema, like the local max at -1 and the local min at 1.

Definition: A point (a, b) in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = [0, 0]$.

13.2. The **Fermat principle** which predates the discovery of calculus tells:

If $\nabla f(x, y)$ is not zero, then (x, y) is not a critical point.

13.3. Proof. Take the directional derivative in the direction $v = \nabla f/|\nabla f|$. Then $D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f| > 0$. You can also see it with linear approximation. If $\nabla f(x_0, y_0) \neq 0$, then the linear approximation $L(x, y)$ is not constant and f is neither maximal nor minimal at (x_0, y_0). QED.

13.4. Note that in our definition, we do **not** include points, where f or its derivative is not defined. Without stating otherwise, we always assume that a function can be differentiated arbitrarily often. Points where the function has no derivatives are not considered part of the domain and need to be studied separately. For the continuous function $f(x, y) = 1/\log(|xy|)$ for example, we would have to look at the points on the coordinate axes as well as the points $xy = 1$ separately.

13.5. In one dimension, we used the condition $f'(x) = 0, f''(x) > 0$ to get a local minimum and $f'(x) = 0, f''(x) < 0$ to assure a local maximum. If $f'(x) = 0, f''(x) = 0$, the nature of the critical point is undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat **inflection point** like for $f(x) = x^3$.

Definition: If $f(x, y)$ is a function of two variables with a critical point (a, b), the number $D = f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** of the critical point.

13.6. The discriminant can be remembered better if seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. As of default, we always assume that functions are twice continuously differentiable. Here is the **second derivative test**:

Theorem: Assume (a, b) is a critical point for $f(x, y)$.
If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.
If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.
If $D < 0$ then (a, b) is a saddle point.

13.7. If $D \neq 0$ at all critical points, the function f is called **Morse**. The Morse condition is nice as for $D = 0$, we need higher derivatives or ad-hoc methods to determine the nature of the critical point.

13.8. To determine the maximum or minimum of $f(x, y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see in the next unit how to get extrema on the boundary.

13.9. Sometimes, we want to find the overall maximum and not only the local ones.

Definition: A point (a, b) in the plane is called a **global maximum** of $f(x, y)$ if $f(x, y) \leq f(a, b)$ for all (x, y). For example, the point $(0, 0)$ is a global maximum of the function $f(x, y) = 1 - x^2 - y^2$. Similarly, we call (a, b) a **global minimum**, if $f(x, y) \geq f(a, b)$ for all (x, y).

Examples

13.10. Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x, y) = [4(x^3 - y), 4(y^3 - x)]$ with critical points $(0, 0), (1, 1), (-1, -1)$.

13.11. $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = [2x\cos(x^2 + y), \cos(x^2 + y) + 1]$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\ldots (0, -\pi), (0, \pi), (0, 3\pi), \ldots$. There are infinitely many.

13.12. The graph of $f(x, y) = (x^2 + y^2)e^{-x^2 - y^2}$ looks like a volcano. The gradient $\nabla f = [2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2)]e^{-x^2 - y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$. This function has a continuum of critical points.

13.13. The function $f(x, y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = [(0, 0]$ for

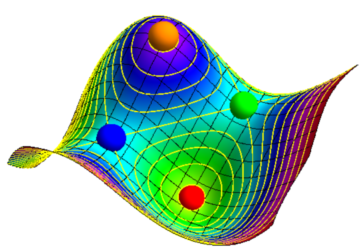
$$(x, y) = \ldots, (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \ldots .$$

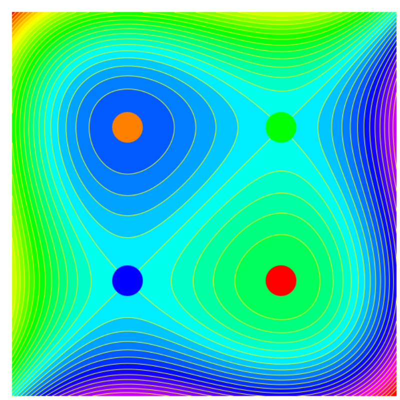
These points are equilibrium points, the angles for which the pendulum is at rest.

13.14. The function $f(x,y) = a\log(y) - by + c\log(x) - dx$ is a function which is invariant by the flow of the **Volterra-Lodka** differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point of f and an equilibrium point of the system.

13.15. The function $f(x,y) = |x| + |y|$ is smooth on the first quadrant $\{x > 0, y > 0\}$. It does not have critical points there. The function has a minimum at $(0,0)$ but it is not in the domain, where f and ∇f are defined. We have to look at the points on the coordinate axis separately. For $y = 0$, we see that $x = 0$ is a minimum. For $x = 0$ we see that $y = 0$ is a minimum. Indeed $(0,0)$ is a minimum of f. This minimum was not detected using derivatives.

13.16. The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x,y) = [x^2 - 1, -y^2 + 1]$. There are 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1,1)$ we have $D = -4$ and so a saddle point,
For $(-1,1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1,-1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1,-1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





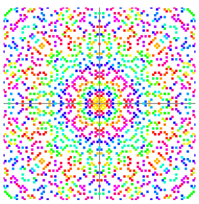
13.17. Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and 4/3 is a local maximum on the line $y = -1$. Comparing $f(4/3,0), f(4/3,-1)$ shows that $(4/3, 0)$ is the global maximum.

13.18. Find the global maxima and minima of $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2$ **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x,0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x,y) = (x^2-1)^2 + (y-1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

Homework

This homework is due on Tuesday, 7/21/2020.

Problem 13.1: Find all the extrema of the function $f(x,y) = xy + x^2y + xy^2$ and determine whether they are maxima, minima or saddle points.



This is a **Gaussian Goldbach function** $f_n = \sum_{k+im \text{prime}, k,m \leq n} x^k y^m$. For $n = 2$ we sum over the Gaussian primes $1+i, 2+i, 1+2i$. (A complex integer $a+ib$ with $a,b \in \mathbb{N}$ is **prime** if a^2+b^2 is a usual prime. A 2D **Goldbach conjecture** claims that all partial derivatives $g^{(p,q)}(0,0)$ with $1 < p,q \leq n$ of $g(x,y) = f_{2n}^2(x,y)$ at $(0,0)$ are non-zero if $p+q$ is even. Equivalently, every Gaussian integer $a+ib$ with $a+b$ even and $a > 1, b > 1$ is a sum of two Gaussian primes in $Q = \{a+ib \mid a > 0, b > 0\}$.

Problem 13.2: Where on the parametrized surface $\vec{r}(u,v) = [1+u^3, v^2, uv]$ is the temperature $T(x,y,z) = 24y - 24z + 2x + 10$ minimal? To find the minimum, look where the function $f(u,v) = T(\vec{r}(u,v))$ has an extremum. Find all local maxima, local minima or saddle points of f.

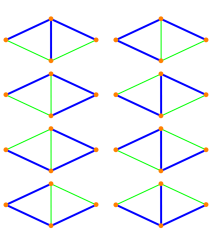
Problem 13.3: Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$.

Problem 13.4: Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 13$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

Problem 13.5: Graph theorists look at the **Tutte polynomial** $f(x,y)$ of a network. We work with the Tutte polynomial

$$f(x,y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**. Classify using the second derivative test.



Remark. The polynomial is useful: $xf(1-x,0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1,1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1,1) = 8$ as you see the 8 possible trees.