

# MULTIVARIABLE CALCULUS

MATH S-21A

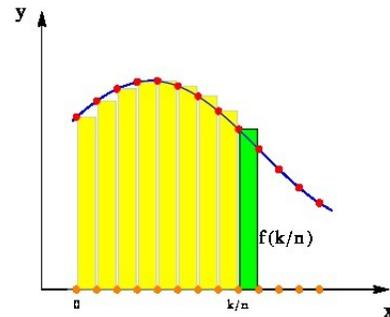
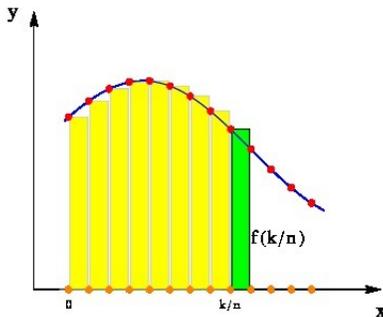
## Unit 15: Double Integrals

### LECTURE

**15.1.** If  $f(x)$  is a continuous function, the **Riemann integral**  $\int_a^b f(x) dx$  is defined as the limit of the **Riemann sums**  $S_n f(x) = \frac{1}{n} \sum_{k/n \in [a,b]} f(k/n)$  for  $n \rightarrow \infty$ . The **derivative** of  $f$  is the limit of **difference quotients**  $D_n f(x) = n[f(x + 1/n) - f(x)]$  as  $n \rightarrow \infty$ . The integral  $\int_a^b f(x) dx$  is the **signed area** under the graph of  $f$  and above the  $x$ -axes, where "signed" indicates that area below the  $x$ -axes has negative sign. The function  $F(x) = \int_0^x f(y) dy$  is called an **anti-derivative** of  $f$ . It is determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x) = F(x) - F(0) .$$

It allows to compute integrals by inverting differentiation so that **differentiation rules** become **integration rules**: the product rule leads to integration by parts, the chain rule becomes partial integration.



**Definition:** If  $f(x, y)$  is continuous on a region  $R$ , the integral  $\iint_R f(x, y) dx dy$  is defined as the limit of the Riemann sum

$$\frac{1}{n^2} \sum_{(\frac{i}{n}, \frac{j}{n}) \in R} f(\frac{i}{n}, \frac{j}{n})$$

when  $n \rightarrow \infty$ . We write also  $\iint_R f(x, y) dA$ , where  $dA = dx dy$  is a notation standing for "an area element".

**15.2. Fubini's theorem** allows to switch the order of integration over a rectangle if the function  $f$  is continuous:

**Theorem:**  $\int_a^b \int_c^d f(x, y) \, dx dy = \int_c^d \int_a^b f(x, y) \, dy dx.$

Proof. For every  $n$ , there is the "quantum Fubini identity"

$$\sum_{\frac{i}{n} \in [a,b]} \sum_{\frac{j}{n} \in [c,d]} f\left(\frac{i}{n}, \frac{j}{n}\right) = \sum_{\frac{j}{n} \in [c,d]} \sum_{\frac{i}{n} \in [a,b]} f\left(\frac{i}{n}, \frac{j}{n}\right)$$

holding for all functions. Now divide both sides by  $n^2$  and take the limit  $n \rightarrow \infty$ . This is possible for continuous functions. Fubini's theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

**Definition:** A **type I region** is of the form

$$R = \{(x, y) \mid a \leq x \leq b, c(x) \leq y \leq d(x)\} .$$

An integral over a type I region is called a **type I integral**

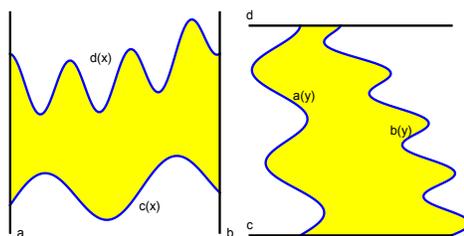
$$\iint_R f \, dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) \, dy dx .$$

A **type II region** is of the form

$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\} .$$

An integral over such a region is called a **type II integral**

$$\iint_R f \, dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) \, dx dy .$$



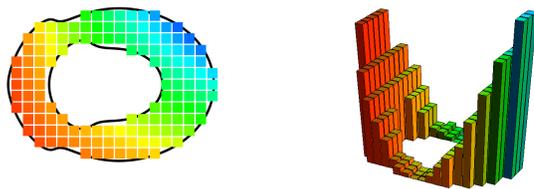
**15.3.** Similarly as we could see in one dimensions an integral as a signed area, one can interpret  $\int \int_R f(x, y) \, dy dx$  as the **signed volume** of the solid below the graph of  $f$  and above  $R$  in the  $xy$  plane. As in 1D integration, the volume of the solid below the  $xy$ -plane is counted negatively.

### EXAMPLES

**15.4.** If we integrate  $f(x, y) = xy$  over the unit square we can sum up the Riemann sum for fixed  $y = j/n$  and get  $y/2$ . Now perform the integral over  $y$  to get  $1/4$ . This example shows how to reduce double integrals to single variable integrals.

**15.5.** If  $f(x, y) = 1$ , then the integral is the **area** of the region  $R$ . The integral is the limit  $L(n)/n^2$ , where  $L(n)$  is the number of lattice points  $(i/n, j/n)$  contained in  $R$ .

**15.6.** The value  $\iint_R f(x, y) dA / \iint_R 1 dA$  is the **average** value of  $f$ .



**15.7.** Integrate  $f(x, y) = x^2$  over the region bounded above by  $\sin(x^3)$  and bounded below by the graph of  $-\sin(x^3)$  for  $0 \leq x \leq \pi$ . The value of this integral has a physical meaning. It is called **moment of inertia**.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2 dy dx = 2 \int_0^{\pi^{1/3}} \sin(x^3) x^2 dx$$

We have now an integral, which we can solve by substitution  $-\frac{2}{3} \cos(x^3) \Big|_0^{\pi^{1/3}} = \frac{4}{3}$ .

**15.8.** Integrate  $f(x, y) = y^2$  over the region bound by the  $x$ -axes, the lines  $y = x + 1$  and  $y = 1 - x$ . The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of  $y$ : they are  $x = y - 1$  and  $x = 1 - y$

$$\int_0^1 \int_{x-1}^{1-x} y^2 dy dx = 1/6 .$$

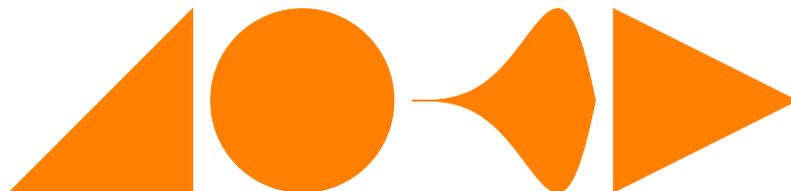
**15.9.** Let  $R$  be the triangle  $1 \geq x \geq 0, 0 \leq y \leq x$ . What is

$$\int \int_R e^{-x^2} dx dy ?$$

The type II integral  $\int_0^1 [\int_y^1 e^{-x^2} dx] dy$  can not be solved because  $e^{-x^2}$  has no anti-derivative in terms of elementary functions.

The type I integral  $\int_0^1 [\int_0^x e^{-x^2} dy] dx$  however can be solved:

$$= \int_0^1 x e^{-x^2} dx = -\frac{e^{-x^2}}{2} \Big|_0^1 = \frac{(1 - e^{-1})}{2} = 0.316... .$$



**15.10.** The area of a disc of radius  $R$  is  $\int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 dy dx = \int_{-R}^R 2\sqrt{R^2 - x^2} dx$ . Substitute  $x = R \sin(u), dx = R \cos(u)$ , to get  $\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2 \sin^2(u)} R \cos(u) du = \int_{-\pi/2}^{\pi/2} 2R^2 \cos^2(u) du = R^2 \pi$ .

## HOMEWORK

This homework is due on Tuesday, 7/21/2020.

**Problem 15.1:** a) (4 points) Find the iterated integral

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2 + (y^2/2)} dy dx .$$

b) (4 points) Now compute

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2 + y^2/2} dx dy .$$

c) (2 points) Wouldn't Fubini assure that a) and b) are the same? What change would be needed in b) to make the results agree?

**Problem 15.2:** Find the area of the region

$$R = \{(x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value  $\int \int_R f(x, y) dx dy / \text{area}(R)$  of  $f(x, y) = y$  over that region.

**Problem 15.3:** Find the volume of the solid lying under the paraboloid  $z = 3x^2 + 3y^2$  and above the rectangle  $R = [-2, 2] \times [-2, 3] = \{(x, y) \mid -2 \leq x \leq 2, -2 \leq y \leq 3\}$ .

**Problem 15.4:** Calculate the iterated integral  $\int_0^1 \int_x^{2-x} (x^2 - y) dy dx$ . Sketch the corresponding bottom to top region. Write this integral as integral over left to right region and compute the integral again.

**Problem 15.5:** There is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{3x^7}{\sqrt{x} - x^2} dx dy .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} 11(x^2 + y^2)^{10} dx dy .$$

You might want to "time travel" one lecture forward, where polar coordinates are known to solve this problem.