MULTIVARIABLE CALCULUS

MATH S-21A

Unit 16: Surface Integration

Lecture

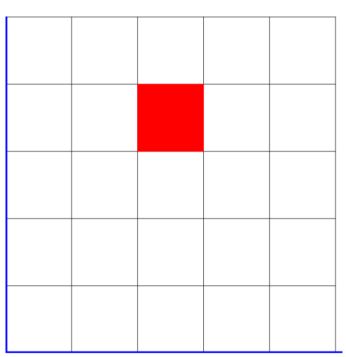
16.1. For certain regions, it is better to use different coordinate system. A reparametrization $(x, y) = \vec{r}(u, v)$ often helps. This works then also in higher dimensions, when suraces are parametrized as $(x, y, z) = \vec{r}(u, v)$. But first to the two dimensional case, where polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta)$ are an important example

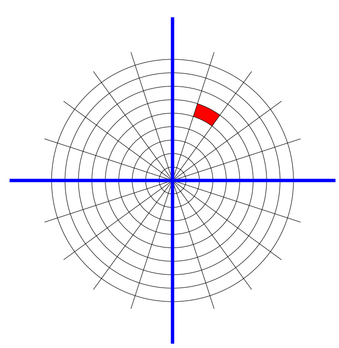
Definition: A **polar region** is a planar region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$. The most common case is $\theta(t) = t$. In Cartesian coordinates the parametrization of the boundary of a polar region is $\vec{r}(t) = [r(t)\cos(\theta(t), r(t)\sin(\theta(t)]$, a **polar graph** like the spiral with $\theta(t) = t$.

Theorem: To integrate in polar coordinates, we evaluate the integral

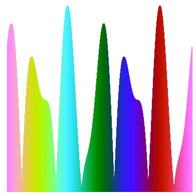
$$\iint_R f(x, y)\, dxdy = \iint_R f(r\cos(\theta), r\sin(\theta) r\; drd\theta$$

16.2. Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x, y) plane. It has the area $r\ d\theta dr$. If you have seen some linear algebra, note that the Jacobean matrix dT has the determinant r.

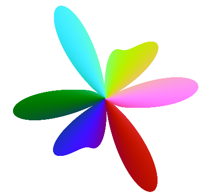




16.3. We can now integrate over type I or type II regions in the (θ, r) plane. like **flowers**: $\{(\theta, r) \,|0 \le r \le f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



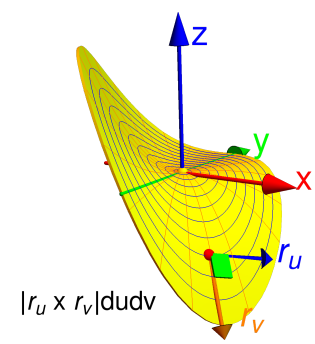
A polar region shown in polar coordinates. It is a type I region.



The same region in the xy coordinate system is not type I or II.

> **Theorem:** A surface $\vec{r}(u, v)$ parametrized on a parameter domain R has the **surface area**
> $$\int\int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u, v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u, v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u + du] \times [v, v + dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$ which has the area $|\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv$.

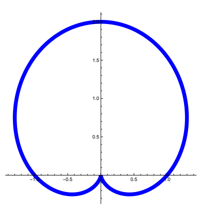


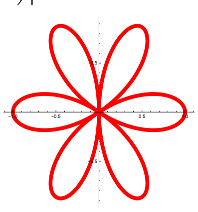
Examples

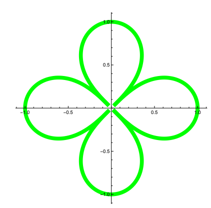
16.4. The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

16.5. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It belongs to the class of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

16.6. The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**.







16.7. Integrate

$$f(x,y) = x^2 + y^2 + xy \ ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\int\int_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta)) r\ d\theta dr = 2\pi/4$.

16.8. We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\ \}$ using substitution. It is more elegant to do this integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r\ drd\theta = 2\pi r^2/2|_0^1 = \pi \ .$$

16.9. Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta)) \mid r(\theta) \leq \ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2 \ .$$

16.10. Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y) \mid 1 < x^2 + y^2 < 4, y > 0\ \}$.

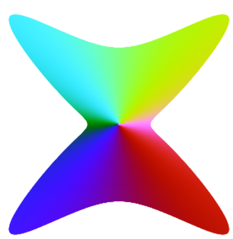
$$\int_1^2 \int_0^\pi r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta)\ d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

16.11. The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This so called **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



16.12. The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

[1]"Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

16.13. The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]$ maps the rectangle $G = [0,2\pi] \times [0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \, dS = \int_0^{2\pi} \int_0^{\pi} L^2 \sin(v) \, dvdu = 4\pi L^2$

16.14. For graphs $(u,v) \mapsto [u,v,f(u,v)]$, we have $\vec{r}_u = (1,0,f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1+f_u^2+f_v^2} \, dudv$.

16.15. Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]$ on $R = [0,2\pi]\times[a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \, dudv = 2\pi \int_a^b |f(v)|\sqrt{1+f'(v)^2} \, dv$.

Homework

This homework is due on Tuesday, 7/21/2020.

Problem 16.1: a) A city near the sea is modeled by a half disk $R = \{(x,y) \mid x^2+y^2 \leq 64, x \geq 0$ with center the origin and radius 8. What is the average distance of a point in D to the origin? in other words, what is the integral $\int\int_R \sqrt{x^2+y^2} \, dxdy / \int\int_D 1 \, dxdy$.
b) The distance to the beach is x. Find the average distance $\iint_R x \, dxdy / \int\int_D 1 \, dxdy$ to the beach.

Problem 16.2: Find $\int\int_R (x^2+y^2)^{40} \, dA$, where R is the part of the unit disc $\{x^2+y^2 \leq 1\}$ for which $y > x$.

Problem 16.3: What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0,2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0,2\pi]$ and third by the positive x-axis?

Problem 16.4: The average of a function f on a region is defined as

$$\frac{\int_R f \, dxdy}{\int_R 1 \, dxdy} .$$

Find the average value of $f(x,y) = 2(x^2+y^2)$ on the annular region $R : 1 \leq |(x,y)| \leq 2$.

Problem 16.5: Find the surface area of the part of the paraboloid $x = y^2+z^2$ which is inside the cylinder $y^2+z^2 \leq 16$.

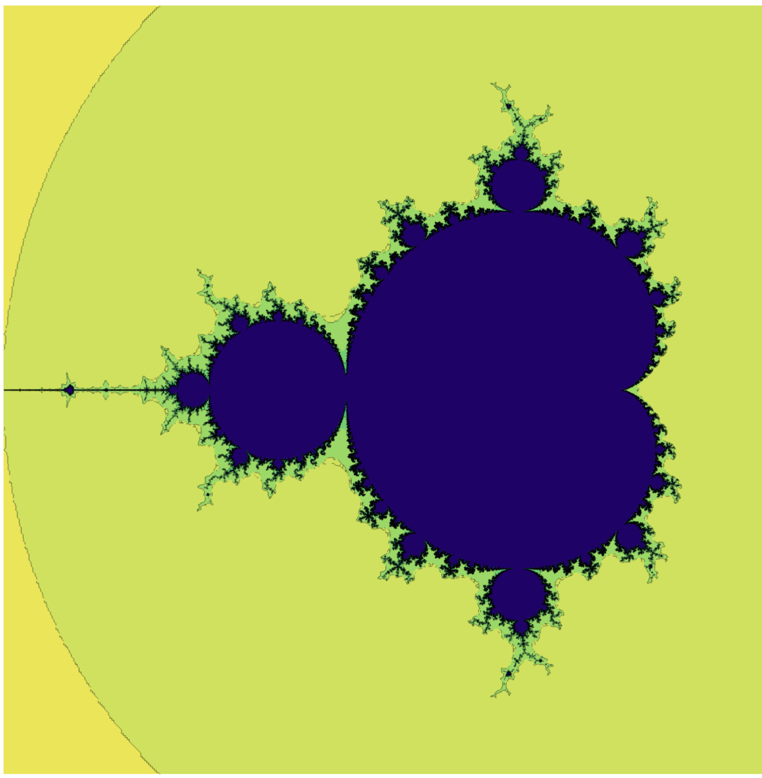
Postscript: Area of the Mandelbrot set

16.16. Often, when we deal with real data, we do not have analytic expressions for the region or function we want to integrate. We want to elaborate here on an example mentioned already in the text. It is the problem to find the area of Mandelbrot set

$$M = \{c = a + ib \in \mathbb{C} \in \mathbb{R}^2 \mid T_c(0)^n \text{ stays bounded } \} ,$$

where $T_c(z) = z^+c$ (as complex numbers, which is written out in real coordinates the map $T_c(x, y) = (x^2 - y^2 + a, 2xy + b)$.

16.17. Here is a picture: it can also be visualized as a function which is 1 on the Mandelbrot set and 0 else.



16.18. What is the area of the Mandelbrot set? We know it is contained in the rectangle $x \in [-2, 1]$ and $y \in [-3/2, 3/2]$. We now just randomly shoot into this rectangle and see whether we are in the Mandelbrot set or not after 1000 iterations. Here is some Mathematica code which allows you to compute things. When we ran it, it gave a value of about 1.515.... More accurate measurements reported hint for a slightly smaller value like 1.506.... Others have given bounds $[1.50311, 1.5613027]$.

```
M=Compile[{x,y},Module[{z=x+I y,k=0},
     While[Abs[z]<2.&&k<1000,z=N[z^2+x+I y];++k];Floor[k/1000]]];
9*Sum[M[-2+3 Random[],-1.5+3 Random[]],{1000000}]/1000000
```

How accurately can you compute the area of the Mandelbrot set? It is a data problem unless somebody comes up with a formula.

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