

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 17: Triple integrals

### LECTURE

**17.1.** Integrating over higher dimensional regions is done in the same way than in two dimensions. Three dimensional regions are referred to as **solids**.

**Definition:** If  $f(x, y, z)$  is continuous and  $E$  is a **bounded solid** in  $\mathbb{R}^3$ , then  $\iiint_E f(x, y, z) \, dx dy dz$  is defined as the  $n \rightarrow \infty$  limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \in E} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right).$$

Triple integrals can be evaluated by iterated single integrals:

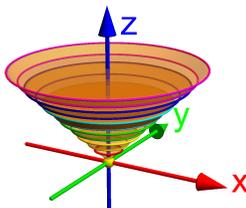
**17.2.** If  $E$  is the box  $\{x \in [1, 2], y \in [0, 1], z \in [0, 1]\}$  and  $f(x, y, z) = 24x^2y^3z$ .

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z \, dz \, dy \, dx.$$

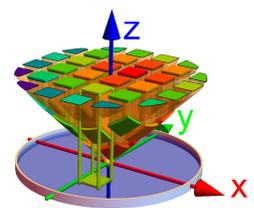
To evaluate the integral, start from the inside  $\int_0^1 24x^2y^3z \, dz = 12x^3y^3$ , then then integrate the middle layer,  $\int_0^1 12x^3y^3 \, dy = 3x^2$  and finally and finally handle the most outer layer:  $\int_1^2 3x^2 \, dx = 7$ .

For the inner integral,  $x = x_0$  and  $y = y_0$  are fixed. The middle integral now computes the contribution over a slice  $z = z_0$  intersected with  $R$ . The outer integral sums up all these slice contributions.

**17.3.** There are two reductions possible to compute triple integrals:



The **burger method** slices the solid a line and computes  $\int_a^b \iint_{R(z)} f(x, y, z) \, dA \, dz$ , where  $g(z)$  is a double integral giving the values when integrating over cheese, meat or tomato. The **fries method** eats up fries going from  $g(x, y)$  to  $h(x, y)$  over a region  $R$ . We have  $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x, y, z) \, dz] \, dA$ .



**17.4.** A special case is the **signed volume**

$$\int \int_R \int_0^{f(x,y)} 1 \, dz dx dy .$$

below the graph of a function  $f(x, y)$  and above a region  $R$ , considered part of the  $xy$ -plane. It is the integral  $\int \int_R f(x, y) \, dA$ . The triple integral which is more natural when considering physical units as volume is measured in cubic meters for example. The triple integral also allows for flexibility: we can replace 1 with a function  $f(x, y, z)$ . If interpreted as a **charge density**, then the integral is the total charge.

**17.5.** The problem of computing volumes has been tackled early. **Archimedes (287-212 BC)** already developed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the **washer method** to reduce the problem to a single variable problem. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. A second method, the **displacement method** is a **comparison technique**: the area of a sphere is the area of the cylinder enclosing it. The volume of a sphere is the volume of the complement of a cone in that cylinder. Modern rearrangement techniques use this still today in modern analysis. Heureka! **Cavalieri (1598-1647)** would build on Archimedes ideas and determine area and volume using tricks now called the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius  $R$ , cut away a cone of height and radius  $R$  from a cylinder of height  $R$  and radius  $R$ . At height  $z$ , this body has a cross section with area  $R^2\pi - r^2\pi$ . If we cut the half sphere at height  $z$ , we obtain a disc of area  $(R^2 - r^2)\pi$ . Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone:  $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$  and the volume of the sphere is  $4\pi R^3/3$ . **Newton (1643-1727)** and **Leibniz (1646-1716)** developed calculus independently. It provided a new tool which made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools. We can do this also in higher dimensions.

#### EXAMPLES

**17.6.** Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions obtained by solving for  $z$ . Let  $R$  be the unit disc in the  $xy$  plane. If we use the **sandwich method**, we get

$$V = \int \int_R \left[ \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz \right] dA .$$

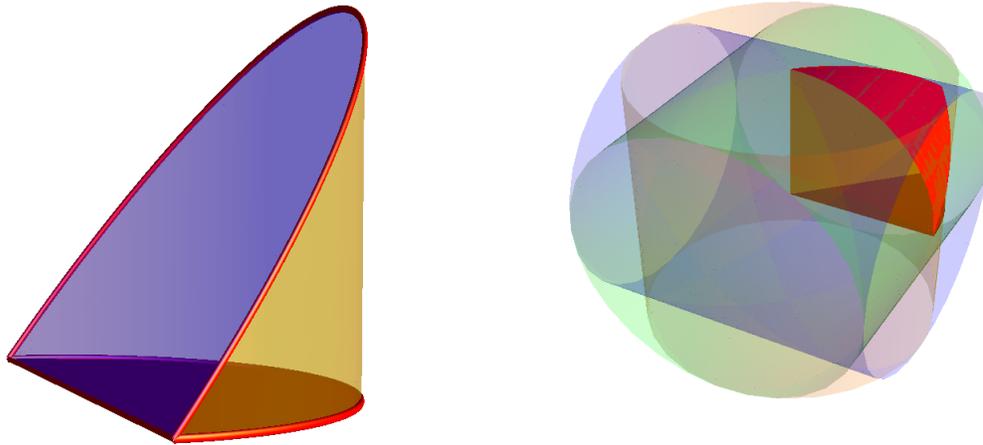
which gives a double integral  $\int \int_R 2\sqrt{1-x^2-y^2} \, dA$  which is of course best solved in polar coordinates. We have  $\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r \, dr d\theta = 4\pi/3$ .

With the **washer method** which is in this case also called **disc method**, we slice along the  $z$  axes and get a disc of radius  $\sqrt{1-z^2}$  with area  $\pi(1-z^2)$ . This is a method suitable for single variable calculus because we get directly  $\int_{-1}^1 \pi(1-z^2) \, dz = 4\pi/3$ .

**17.7.** The mass of a body with mass density  $\rho(x, y, z)$  is defined as  $\int \int \int_R \rho(x, y, z) dV$ . For bodies with constant density  $\rho$ , the mass is  $\rho V$ , where  $V$  is the volume. Compute the mass of a body which is bounded by the parabolic cylinder  $z = 4 - x^2$ , and the planes  $x = 0, y = 0, y = 6, z = 0$  if the density of the body is  $z$ . **Solution:**

$$\begin{aligned} \int_0^2 \int_0^6 \int_0^{4-x^2} z dz dy dx &= \int_0^2 \int_0^6 (4-x^2)^2/2 dy dx \\ &= 6 \int_0^2 (4-x^2)^2/2 dx = 6 \left( \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right) \Big|_0^2 = 2 \cdot 512/5 \end{aligned}$$

**17.8.** The solid region bound by  $x^2 + y^2 = 1, x = z$  and  $z = 0$  is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed a Riemann sum integration technique. It appears in every calculus text book. Find the volume of the hoof. **Solution.** Look from the situation from above and picture it in the  $x - y$  plane. You see a half disc  $R$ . It is the floor of the solid. The roof is the function  $z = x$ . We have to integrate  $\int \int_R x dx dy$ . We got a double integral problems which is best done in polar coordinates;  $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta) dr d\theta = 2/3$ .



**17.9.** Finding the volume of the solid region bound by the three cylinders  $x^2 + y^2 = 1, x^2 + z^2 = 1$  and  $y^2 + z^2 = 1$  is one of the most famous volume integration problems going back to Archimedes.

**Solution:** look at  $1/16$ 'th of the body given in cylindrical coordinates  $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$ . The roof is  $z = \sqrt{1-x^2}$  because above the "one eighth disc"  $R$  only the cylinder  $x^2 + z^2 = 1$  matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1-r^2 \cos^2(\theta)} r dr d\theta$$

has an inner  $r$ -integral of  $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$ . Integrating this over  $\theta$  can be done by integrating  $(1 + \sin(x)^3)\sec^2(x)$  by parts using  $\tan'(x) = \sec^2(x)$  leading to the anti derivative  $\cos(x) + \sec(x) + \tan(x)$ . The result is  $16 - 8\sqrt{2}$ .

HOMEWORK

This homework is due on Tuesday, 7/28/2020.

**Problem 17.1:** Evaluate the triple integral

$$\int_0^3 \int_0^z \int_0^{4y} e^{-y^2} z \, dx dy dz .$$

**Problem 17.2:** Find the volume of the solid bounded by the paraboloids  $z = x^2 + y^2$  and  $z = 36 - (x^2 + y^2)$  and satisfying  $x \geq 0, y \geq 0$ .

**Problem 17.3:** Find the **moment of inertia**  $\int \int \int_E (x^2 + y^2) \, dV$  of a cone

$$E = \{x^2 + y^2 \leq z^2 \ 0 \leq z \leq 15\} ,$$

which has the  $z$ -axis as its center of symmetry.

**Problem 17.4:** Integrate  $f(x, y, z) = x^2 + y^2 - z$  over the tetrahedron with vertices  $(0, 0, 0), (4, 4, 0), (0, 4, 0), (0, 0, 12)$ .

**Problem 17.5:** This is a classic problem of Archimedes: What is the volume of the body obtained by intersecting the solid cylinders  $x^2 + z^2 \leq 9$  and  $y^2 + z^2 \leq 9$ ?

