

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 19: Vector fields

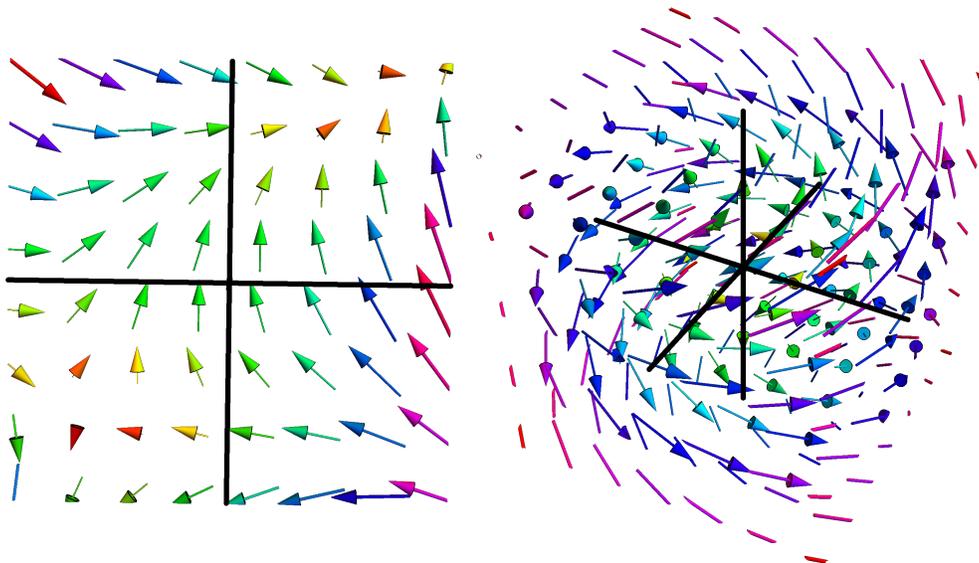
### LECTURE

**17.1.** We have already seen geometric objects like curves, surfaces or solids. Then we have seen functions, which we consider to be scalar fields. If the function becomes vector valued we have **vector field**.

**Definition:** A **planar vector field** is a vector-valued map  $\vec{F}$  which assigns to a point  $(x, y) \in \mathbb{R}^2$  a vector  $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ . A **vector field in space** is a map, which assigns to each point  $(x, y, z) \in \mathbb{R}^3$  a vector  $\vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]$ .

**17.2.** Here are examples of vector fields in two and three dimensions

$$\vec{F}(x, y) = \begin{bmatrix} y - \sin(x) \\ x^3 + \cos(2y) \end{bmatrix}, \vec{F}(x, y, z) = \begin{bmatrix} -y \\ x \\ \sin(z) \end{bmatrix}.$$



**Definition:** If  $f(x, y)$  is a function of two variables, then  $\vec{F}(x, y) = \nabla f(x, y)$  is called a **gradient field**. Gradient fields in space are of the form  $\vec{F}(x, y, z) = \nabla f(x, y, z)$ . They are important!

**17.3.** When is a vector field a gradient field?  $\vec{F}(x, y) = [P(x, y), Q(x, y)] = \nabla f(x, y)$  implies  $Q_x(x, y) = P_y(x, y)$ . If this does not hold at some point,  $\vec{F}$  is no gradient field.

**Clairaut test:** If  $Q_x(x, y) - P_y(x, y)$  is not zero at some point, then  $\vec{F}(x, y) = [P(x, y), Q(x, y)]$  is not a gradient field.

**17.4.** We will see next week that  $\text{curl}(\vec{F}) = Q_x - P_y = 0$  is also sufficient for  $\vec{F}$  to be a gradient field if  $\vec{F}$  is defined everywhere. How do we get  $f$  the function with  $\vec{F} = \nabla f$ ? We will look at examples in class.

#### EXAMPLES

**17.5.** Is the vector field  $\vec{F}(x, y) = [P, Q] = [3x^2y + y + 2, x^3 + x - 1]$  a gradient field? **Solution:** the Clairaut test shows  $Q_x - P_y = 0$ . We integrate the equation  $f_x = P = 3x^2y + y + 2$  and get  $f(x, y) = 2x + xy + x^3y + c(y)$ . Now take the derivative of this with respect to  $y$  to get  $x + x^3 + c'(y)$  and compare with  $x^3 + x - 1$ . We see  $c'(y) = -1$  and so  $c(y) = -y + c$ . We see the solution  $\boxed{x^3y + xy - y + 2x}$ .

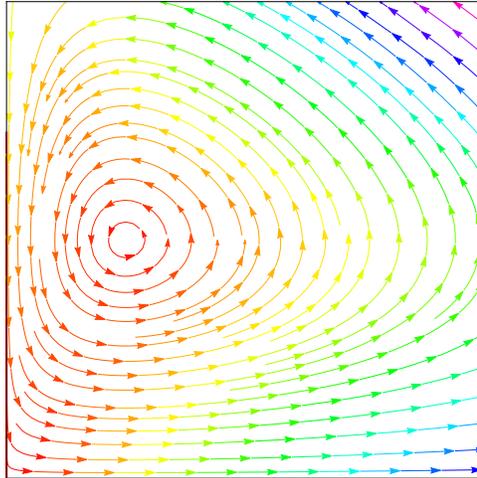
**17.6.** Is the vector field  $\vec{F}(x, y) = [xy, 2xy^2]$  a gradient field? **Solution:** No:  $Q_x - P_y = 2y^2 - x$  is not zero.

Vector fields appear naturally when studying differential equations. Here is an example in population dynamics:

**17.7.** If  $x(t)$  is the population of a “prey species” like tuna fish and  $y(t)$  is the population size of a “predator” like sharks. We have  $x'(t) = ax(t) - bx(t)y(t)$  with positive  $a, b$  because both more predators and more prey species will lead to prey consumption. The rate of change of  $y(t)$  is  $y'(t) = -cy(t) + dxy$ , where  $c, d$  are positive. This can be written using a vector field  $\vec{r}' = \vec{F}(\vec{r}(t))$ . We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lotka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy, \end{aligned}$$

where  $\vec{F}(x, y) = [0.4x - 0.4xy, -0.1y + 0.2xy]$ . Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point  $\vec{r}(x, y) = [x(t), y(t)]$ , there is a curve  $= \vec{r}(t) = [x(t), y(t)]$  through that point for which the tangent  $\vec{r}'(t) = (x'(t), y'(t))$  is the vector field.

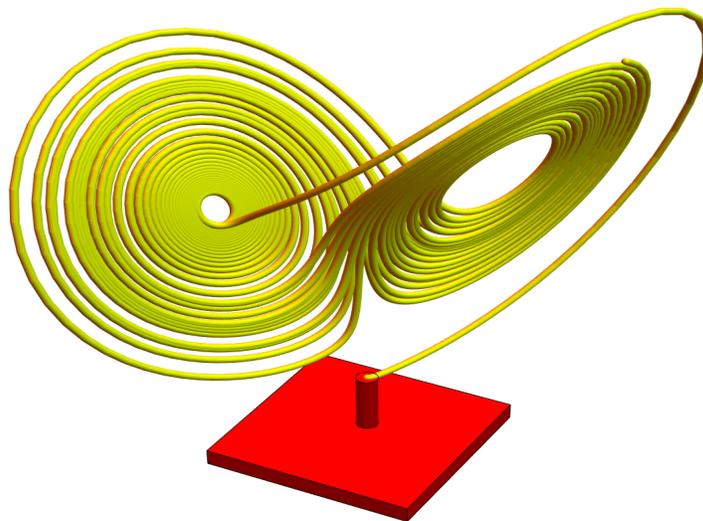


**17.8.** In mechanics the class of **Hamiltonian fields** plays an important role: if  $H(x, y)$  is a function of two variables, then  $[H_y(x, y), -H_x(x, y)]$  is called a **Hamiltonian vector field**. An example is the harmonic oscillator  $H(x, y) = (x^2 + y^2)/2$ . Its vector field  $(H_y(x, y), -H_x(x, y)) = (y, -x)$ . The flow lines of a Hamiltonian vector fields are located on the level curves of  $H$ .

**17.9.** Here is a famous example. It is the **Lorenz vector field**

$$\vec{F}(x, y, z) = \begin{bmatrix} 10y - 10x \\ -xz + 28x - y \\ xy - \frac{8}{3}z \end{bmatrix} .$$

It features a so called **strange attractor**.



## HOMEWORK

This homework is due on Tuesday, 7/28/2020.

**Problem 19.1:**

- a) Draw the gradient vector field of  $f(x, y) = \cos(x^2 - y^2)$ .  
 b) Draw the gradient vector field of  $f(x, y) = (x - 1)^2 + (y - 2)^2$ .  
 In both cases, draw a contour map of  $f$  and use gradients to draw the vector field  $F(x, y) = \nabla f$ .

**Problem 19.2:** The vector field

$$\vec{F}(x, y) = \begin{bmatrix} \frac{x}{(x^2+y^2)^{(3/2)}} \\ \frac{y}{(x^2+y^2)^{(3/2)}} \end{bmatrix}$$

appears in electrostatics. Find a function  $f(x, y)$  such that  $\vec{F} = \nabla f$ .

**Problem 19.3:**

- a) Is the vector field  $\vec{F}(x, y) = \begin{bmatrix} xy \\ x^2 \end{bmatrix}$  a gradient field?  
 b) Is the vector field  $\vec{F}(x, y) = \begin{bmatrix} \sin(x) + y \\ \cos(y) + x \end{bmatrix}$  a gradient field?

In both cases, find  $f(x, y)$  satisfying  $\nabla f(x, y) = \vec{F}(x, y)$  or give a reason, why it does not exist.

**Problem 19.4:** Find conditions such that a vector field  $\vec{F}(x, y, z)$  is a gradient field. Then check it in the following cases. If there is a gradient field, find  $f$  such that  $\vec{F} = \nabla f$ .

- a)  $\vec{F}(x, y, z) = [x^{11}, y^9, z]$ .  
 b)  $\vec{F}(x, y, z) = [y, x, z^3]$ .  
 c)  $\vec{F}(x, y, z) = [10y + 10x, 10x + 10y, x]$ .  
 d)  $\vec{F}(x, y) = [y, z, x]$ .

**Problem 19.5:** Find the potential  $f$  to

$$\vec{F}(x, y, z) = [5e^{5x} + 5x^4y + z^4 + y \cos(xy), x^5 + x \cos(xy), 4xz^3 + 7e^{7z}] .$$