MULTIVARIABLE CALCULUS

MATH S-21A

Unit 22: Curl and Flux

Lecture

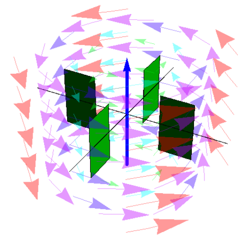
22.1. The **curl** in two dimensions was the scalar field $\text{curl}(F) = Q_x - P_y$. By Green's theorem, the curl evaluated at (x, y) is $\lim_{r\to 0} \int_{C_r} \vec{F}\, dr/(\pi r^2)$, where C_r is a small circle of radius r oriented counter clockwise an centered at (x, y). Green's theorem explains so what the curl is: it measures how the field "curls"... As rotations in two dimensions are determined by a single angle, in three dimensions, three parameters are needed. It is a vector whose direction tells the axes of rotation and the length tells the amount of rotation. The curl now becomes a vector:

Definition: The **curl** of $\vec{F} = [P, Q, R]$ is the vector field
$$\text{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y]\ .$$

22.2. In **Nabla calculus**, this is written as $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component $Q_x - P_y$ of the curl is for fixed z just the curl of the two-dimensional vector field $\vec{F} = [P, Q]$. While the curl in two dimensions is a scalar field, it is a vector field in 3 dimensions. In n dimensions, it would have $n(n-1)/2$ components, as this is the number of 2-dimensional coordinate planes. The curl measures the "vorticity" of the field and each component measures this in one of the two dimensional coordinate planes.

Definition: If a field has zero curl everywhere, the field is called **irrotational**.

22.3. The curl is frequently visualized using a "paddle wheel". If the rotation axes points into direction v, the rotation speed is $\vec{F} \cdot \vec{v}$. The direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. The angular velocity of the wheel is the length of the curl.



22.4. In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives: the **gradient**, the **curl** and the **divergence**.

Definition: The **divergence** of $\vec{F} = [P,Q,R]$ is the scalar field $\mathrm{div}([P,Q,R]) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

22.5. The divergence can also be defined in two dimensions, but it is there not as fundamental as it is not an "exterior derivatives". We want in d dimensions to have d fundamental derivatives and d fundamental integrals and d fundamental theorems. Distinguishing dimensions helps to organize the integral theorems. While Green looks like Stokes, we urge you to look at it as a different theorem taking place in "flatland". It is a small matter but it is much clearer to have in every dimension d a separate calculus. This prevents mixing up the theorems and makes things easier.

Definition: In two dimensions, the **divergence** of $\vec{F} = [P,Q]$ is defined as $\mathrm{div}(P,Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

22.6. In two dimensions, the divergence can be written as the curl of a -90 degrees rotated field $\vec{G} = [Q,-P]$ because $\mathrm{div}(\vec{G}) = Q_x - P_y = \mathrm{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

22.7. With the "vector" $\nabla = [\partial_x, \partial_y, \partial_z]$, we can write $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$. Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the ∇ vector is not an actual vector but an operator. The following combination of divergence and gradient often appears in physics:

Definition:

$$\Delta f = \mathrm{div}(\mathrm{grad}(f)) = f_{xx} + f_{yy} + f_{zz} \ .$$

is called the **Laplacian** of f. One can write $\Delta f = \nabla^2 f$.

22.8. Mathematicians know Δ it as a 'form Laplacian". Here are some identities:

$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0.$
$\mathrm{curlgrad}(\vec{F}) = \vec{0}$
$\mathrm{curl}(\mathrm{curl}(\vec{F})) = \mathrm{grad}(\mathrm{div}(\vec{F}) - \Delta(\vec{F})).$

Examples

22.9. Question: Is there a vector field $\vec{G}$ such that $\vec{F} = [x + y, z, y^2] = \mathrm{curl}(\vec{G})$? **Answer:** No, because $\mathrm{div}(\vec{F}) = 1$ is incompatible with $\mathrm{div}(\mathrm{curl}(\vec{G})) = 0$.

22.10. Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \mathrm{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \mathrm{grad}(f)$. As the region is simply connected, we can deform any path between two points without changing the result. Now, $\mathrm{div}(F) = 0$ implies $\mathrm{divgrad}(f) = \Delta f = 0$.

22.11. If we rotate the vector field $\vec{F} = [P, Q]$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = [-Q, P]$. The integral $\int_C F \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\mathrm{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\mathrm{curl}(\vec{F}) = \mathrm{div}(\vec{G}) \ .$$

Green's theorem now becomes

$$\int \int_R \mathrm{div}(\vec{G}) \ dxdy = \int_C \vec{G} \cdot \vec{dn} \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

In two dimensions, the divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero.

We have now all the derivatives we need. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\mathrm{grad}} 1$$

$$1 \xrightarrow{\mathrm{grad}} 2 \xrightarrow{\mathrm{curl}} 1$$

$$1 \xrightarrow{\mathrm{grad}} 3 \xrightarrow{\mathrm{curl}} 3 \xrightarrow{\mathrm{div}} 1$$

Homework

This homework is due on Tuesday, 8/4/2020.

Problem 22.1: Construct your own nonzero vector field $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ in each of the following cases:
a) $\vec{F}$ is irrotational but not incompressible.
b) $\vec{F}$ is incompressible but not irrotational.
c) $\vec{F}$ is irrotational and incompressible.
d) $\vec{F}$ is not irrotational and not incompressible.

Problem 22.2: The vector field $\vec{F}(x, y, z) = [x, y, -2z]$ satisfies $\mathrm{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x, y, z)$ such that $\mathrm{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $[x, 0, -z] + [0, y, -z]$ and find vector potentials for each of the parts using a vector field you have seen on the blackboard in class.

Problem 22.3: Evaluate the flux integral $\int\int_S [0, 0, yz] \cdot d\vec{S}$, where S is the surface with parametric equation $x = uv, y = u + v, z = u - v$ on $R : u^2 + v^2 \leq 4$ and $u > 0$.

Problem 22.4: Evaluate the flux integral $\int\int_S \mathrm{curl}(F) \cdot d\vec{S}$ for

$$\vec{F}(x, y, z) = [3xy, 3yz, 3zx] \ .$$

where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0, 2] \times [0, 2]$ and has an upward orientation.

Problem 22.5: a) What is the relation between the flux of the vector field $\vec{F} = \nabla g/|\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x, y, z) = x^6 + y^4 + 2z^8$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times [0, 0, 2]$ through the surface S.

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[1] Both a and b) do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot d\vec{S}$ with dS in that case.