

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 23: Stokes Theorem

LECTURE

23.1. We work with a surface S parametrized as $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ over a domain R in the uv -plane. Remember that the **flux integral** of \vec{F} through S is defined as the double integral

$$\iint_R \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv .$$

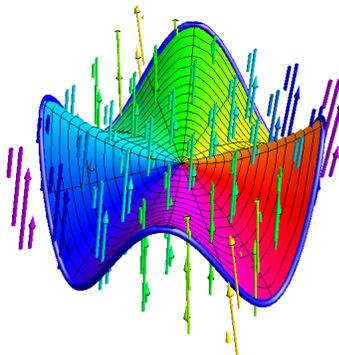
The following theorem is the second fundamental theorem of calculus in three dimensions:

Definition: The **boundary** of a surface S consists of all points P for which even arbitrary small circle $S_r(P) \cap S$ around the point is not closed.

23.2. The boundary is a collection of curves oriented so that the surface is to the “left” if the normal vector to the surface is pointing “up”. In other words, the velocity vector \vec{v} , a vector \vec{w} pointing towards the surface and the normal vector \vec{n} to the surface form a right handed coordinate system.

Theorem: Stokes theorem: Let S be a surface bounded by a curve C and \vec{F} be a vector field. Then

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$



Proof. Stokes theorem is proven in the same way than Green's theorem. Chop up S into a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of S . Stokes theorem for a small triangle can be reduced to Green's theorem because with a coordinate system such that the triangle is in the xy plane, the flux of the field is the double integral of $\text{curl} \vec{F} \cdot \vec{n} \, dudv = (Q_x - P_y) \cos(\theta) \, dudv$, where θ is the angle between the normal vector and $\vec{F} = [P, Q, R]$. On the other hand, since the power $\vec{F}(\vec{r}) \cdot \vec{r}'(t) \, dt = (P(\vec{r}) \cos(\theta) x'(t) + Q(\vec{r}) \cos(\theta) y'(t)) \, dt$ also has everything multiplied by $\cos(\theta)$, the result for each space triangle follows from Green. Stokes theorem now follows by making the triangulation finer and finer. On both sides we have a Riemann sum approximation to the integrals. \square

EXAMPLES

23.3. Let $\vec{F}(x, y, z) = [-y, x, 0]$ and let S be the upper hemisphere, then $\text{curl}(\vec{F})(x, y, z) = [0, 0, 2]$. The surface is parameterized by

$$\vec{r}(u, v) = [\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)]$$

on $R = [0, 2\pi] \times [0, \pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v) \vec{r}'(u, v)$ so that $\text{curl}(\vec{F})(x, y, z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v) \sin(v) 2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \, dv \, du = 2\pi$.

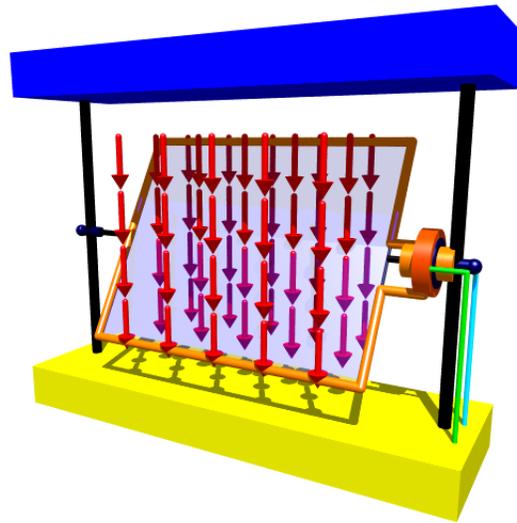
The boundary C of S is parameterized by $\vec{r}(t) = [\cos(t), \sin(t), 0]$ so that $d\vec{r} = \vec{r}'(t) \, dt = [-\sin(t), \cos(t), 0] \, dt$ and $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π .

23.4. If S is a surface in the xy -plane and $\vec{F} = [P, Q, 0]$ has zero z component, then $\text{curl}(\vec{F}) = [0, 0, Q_x - P_y]$ and $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y \, dx \, dy$. We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the xy -plane, then the vector field F induces a vector field on the surface such that its 2D curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F})[R_y - Q_z, P_z - R_x, Q_x - P_y]$ is the two dimensional curl: $\vec{F}(\vec{r}(u, v)) \cdot [0, 0, 1] = Q_x - P_y$. If C is the boundary of the surface, then $\int_S \vec{F}(\vec{r}(u, v)) \cdot [0, 0, 1] \, dudv = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$.

23.5. Calculate the flux of the curl of $\vec{F}(x, y, z) = [-y, x, 0]$ through the surface parameterized by $\vec{r}(u, v) = [\cos(u) \cos(v), \sin(u) \cos(v), \cos^2(v) + \cos(v) \sin^2(u + \pi/2)]$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

23.6. For every surface bounded by a curve C , the flux of $\text{curl}(\vec{F})$ through the surface is the same. Proof. The flux of the curl of a vector field through a surface S depends only on the boundary of S . Compare this with the earlier statement that for every curve between two points A, B the line integral of $\text{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C .

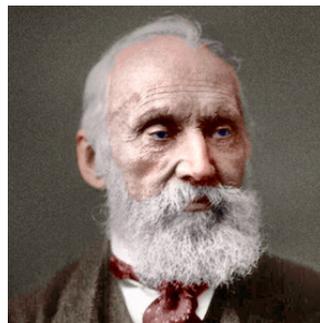
23.7. Electric and magnetic fields are linked by the **Maxwell equation** like $\text{curl}(\vec{E}) = -\frac{1}{c}\dot{\vec{B}}$. These are examples of partial differential equations. If a closed wire C bounds a surface S then $\iint_S \vec{B} \cdot d\vec{S}$ is the flux of the magnetic field through S . Its change can be related with a voltage using Stokes theorem: $d/dt \iint_S \vec{B} \cdot d\vec{S} = \iint_S \dot{\vec{B}} \cdot d\vec{S} = \iint_S -c \text{curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \cdot d\vec{r} = U$, where U is the voltage. If we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator, like the alternator in a car. Stokes theorem explains why we can generate electricity from motion.



23.8. The history of Stokes theorem is a bit hazy. ¹ A version of Stokes theorem appeared to be known by **André Ampère** in 1825. **William Thomson** (Lord Kelvin) mentioned the theorem to Stokes in 1850). **George Gabriel Stokes** (1819-1903) who found parts of the identity earlier 1840 formulated it in a prize exam from 1854 (the proof is one of the exam problems). The first pushed proof is by Hermann Hankel in 1861.



George Gabriel Stokes



William Thomson (Kelvin)



André Marie Ampere

¹See V. Katz, the History of Stokes theorem, Mathematics Magazine 52, 1979, p 146-156

HOMEWORK

This homework is due on Tuesday, 8/4/2020.

Problem 23.1: Assume S is the surface $x^{66} + y^{34} + z^{24} = 1000$ and $\vec{F} = [2 + e^{e^{xyz}}, x^4yz, x - y - \cos(zx)]$. Explain why $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Problem 23.2: Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = [3x^2y, x^3, 3xy]$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 23.3: Evaluate the flux integral $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 23.4: Find the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle of radius 5 in the xz -plane oriented counter clockwise when looking from the point $(0, 1, 0)$ onto the plane and where \vec{F} is the vector field

$$\vec{F}(x, y, z) = [x^2z + x^5, \cos(e^y), -xz^2 + \sin(\sin(z))].$$

Use a convenient surface S which has C as a boundary.

Problem 23.5: Find the flux integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = [2 \cos(\pi y)e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz]$ and S is the surface parametrized by

$$\vec{r}(s, t) = [(1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s]$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.