Unit 2

Vectors Dotproduct

① Vectors

$Q = (3,4,7)$

$(1,1,2) = P$

magnitude

direction

$$\vec{v} = \vec{PQ} = [2,3,5]$$

components

vector

$\vec{w} = [1,1,5]$

$\vec{v} = [2,1,3]$

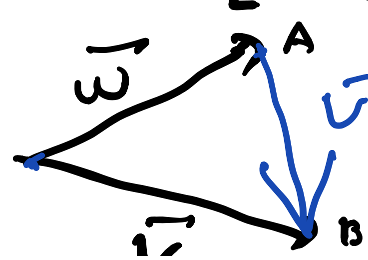
$\lambda(\vec{v}+\vec{w}) = \lambda\vec{v}+\lambda\vec{w}$

$\vec{v}+\vec{w} = \vec{w}+\vec{v}$

$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

add

$$\vec{v}+\vec{w} = [3,2,8]$$



$\boxed{\vec{v}-\vec{w}}$ Subtr

$$\vec{v}-\vec{w} = [1,0,-2]$$

$3\vec{v} = [6, 3, 9]$

Scalar mult.

$3(\vec{v} - \vec{w}) = [3, 0, -6]$

$-\vec{v} = [-2, -1, -3]$

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② $\boxed{\text{Dot product}}$

$[3, 2, 5] \cdot [2, 1, -1]$

$3 \cdot 2 + 2 \cdot 1 + 5 \cdot (-1) = 3$

dot product.

$\vec{u} \cdot \vec{v} = \langle \vec{u} | \vec{v} \rangle$

$= u_i v^i$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2$

$\sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$

magnitude

length

distance

$= d(P, Q)$

$\dfrac{\vec{v}}{|\vec{v}|}$ unit vector = direction

only defined if $\vec{v} \neq 0$

$\vec{v} = \vec{0} = \vec{PP}$ $\quad \dot{P}$

has no direction $\quad \vec{0} + \vec{v} = \vec{v}$

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③ Angles

Theorem (Cauchy-Schwarz)

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}|\,|\vec{w}|$$

Proof: $|\vec{w}| = 1$ WLOG.

$$a = \vec{v} \cdot \vec{w}$$

$$0 \leq |(\vec{v} - a\vec{w})|^2$$

$$= (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w})$$

$$\underset{\text{FOIL}}{=} \vec{v} \cdot \vec{v} + a^2 \vec{w} \cdot \vec{w} - 2a\,\vec{v} \cdot \vec{w}$$

$$= |\vec{v}|^2 + a^2 - 2a^2$$

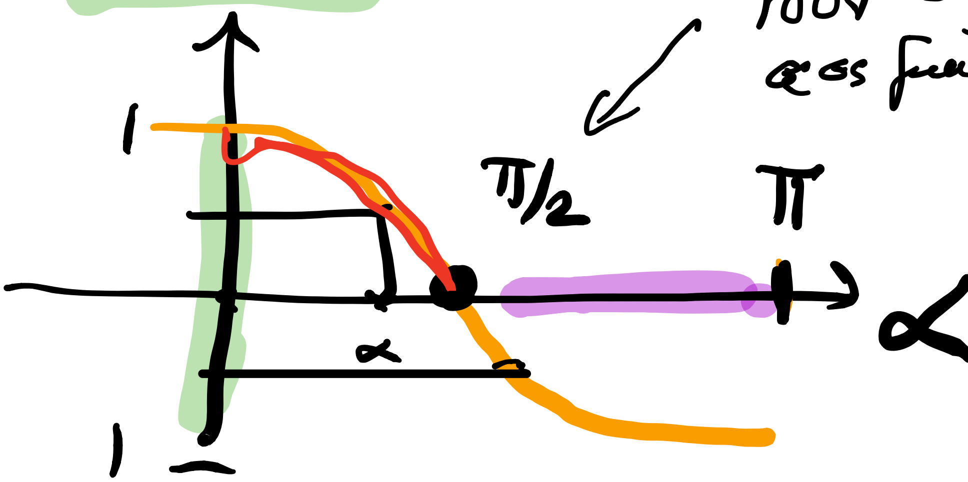
$$= |v|^2 - a^2$$

$a^2 \leq |v|^2 |w|^2$ take square

$(\vec{v} \cdot \vec{w})^2 \leq |v|^2 \, |w|^2$

QED

$$-1 \le \boxed{\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}} \le 1$$



Define α as

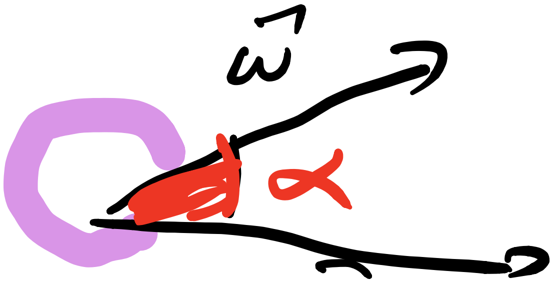
$$\boxed{\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|}}$$

covarian

correlation

$|\vec{v}|$ stand. dev.

$$\vec{v}\cdot\vec{w} = |\vec{v}|\,|\vec{w}|\cdot\cos\alpha$$



$\vec{v}\cdot w > 1$

α is

a calc

angle

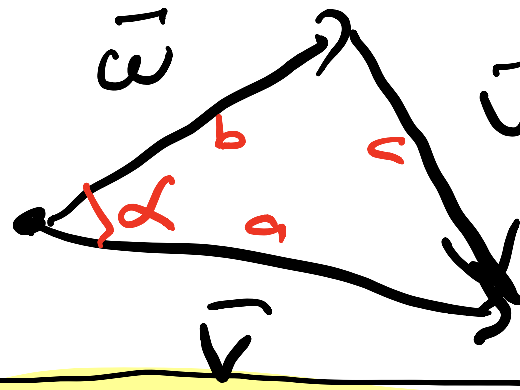
$\vec{v}\cdot\vec{w} < 0$ obtuse

$\vec{v}\cdot\vec{w} = 0$ then

α is $\frac{\pi}{2} = 90^\circ$

right angle.

④ Pythagoras



$|\vec{v}| = a$
$|\vec{w}| = b$
$|\vec{v}-\vec{w}| = c$

Al khashi formula:

$$c^2 = a^2 + b^2 - 2ab\cos\alpha$$

Proof:

$$c^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$

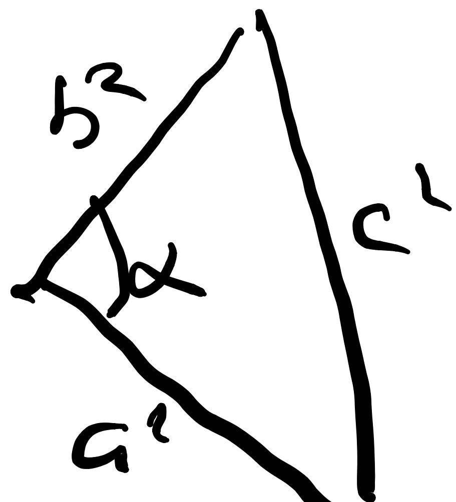
$$\underset{\text{FOIL}}{=} \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\,\vec{v}\cdot\vec{w}$$

$$= |v|^2 + |w|^2 - 2|\vec{v}||\vec{w}|\cos\alpha$$

$$= a^2 + b^2 - 2ab\cos\alpha$$

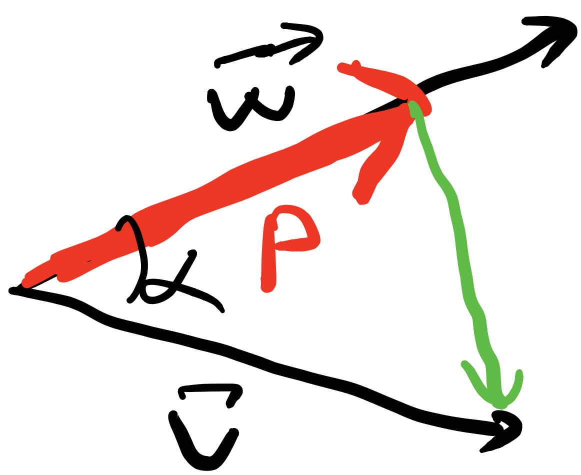
Special case: **QED**

$$\alpha = \frac{\pi}{2}$$



Pythagoras
~600 BC

$$\boxed{c^2 = a^2 + b^2}$$



$P_w v$ = projection onto $\bar{w}$

$$|\vec{p}| = |\vec{v}| \cos\alpha \, \frac{|\vec{w}|}{|\vec{w}|}$$

$$= \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$$

$$\boxed{\begin{aligned} \vec{p} &= \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \cdot \frac{\vec{w}}{|\vec{w}|} \\ &= \frac{\vec{v} \cdot \vec{w}}{|w|^2} \, \vec{w} \end{aligned}}$$

vector project

<u>Example:</u>

$$\vec{v} = [2, 3, 4]$$

$$\vec{w} = [1, 1, 1]$$

$$P_{\vec{w}} \vec{v} = \frac{9}{3} [1, 1, 1]$$

$$= [3, 3, 3]$$