

Unit 3

1) Cross product

$$\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

Diagram illustrating the cross product calculation. The first vector is $\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and the second is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. The result is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$. The calculation is shown with colored lines: blue lines connect the 3 and 1 to the 2, the 4 and 1 to the 1, and the 2 and 1 to the -5. Orange numbers 3, 4, 2 are written below the first vector, and orange numbers 2, 1, 1 are written below the second vector. A purple 'X' is drawn over the second vector.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Diagram illustrating the cross product calculation. The first vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the second is $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. The result is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. The calculation is shown with orange lines: orange lines connect the 1 and 3 to the 1, the 1 and 4 to the -2, and the 1 and 2 to the 1. Orange numbers 1, 1, 1 are written below the first vector, and orange numbers 2, 3, 4 are written below the second vector. A purple 'X' is drawn over the second vector.

2) Properties

(i) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
anticommutative

(ii) $\vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{v} & \vec{w} \\ \hat{i} & \hat{j} & \hat{k} \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$

$\det \begin{pmatrix} v_1 & v_2 & v_3 \\ v_2 w_3 - v_3 w_2 & v_3 w_1 - v_1 w_3 & v_1 w_2 - v_2 w_1 \end{pmatrix}$

$= v_1 v_2 w_3 - v_1 v_3 w_2 + v_2 v_3 w_1 - v_2 v_1 w_3 + v_3 v_1 w_2 - v_3 v_2 w_1$

(candy-crush) = 0

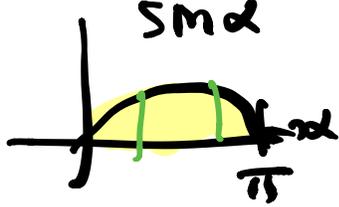
do the same for \vec{w}

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{n(n-1)}{2}$
 $\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1$
 $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$

(iii) $|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin \alpha$

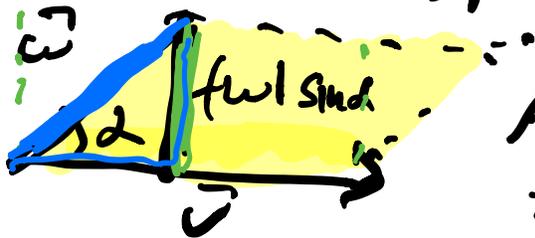
Cauchy-Binet: $|\vec{v}|^2 |\vec{w}|^2 = |\vec{v} \cdot \vec{w}|^2 + |\vec{v} \times \vec{w}|^2$
 \rightarrow in comp

$|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - |\vec{v} \cdot \vec{w}|^2$
 because $|\vec{v} \cdot \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 \cos^2 \alpha$

$\sin \alpha$

 $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 (1 - \cos^2 \alpha)$
 $= |\vec{v}|^2 |\vec{w}|^2 \sin^2 \alpha$
 $|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin \alpha$

QED

(iv) $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by \vec{v} and \vec{w}



$A = |\vec{v}| \text{ height}$
 $= |\vec{v}| |\vec{w}| \sin \alpha$
 $= |\vec{v} \times \vec{w}|$

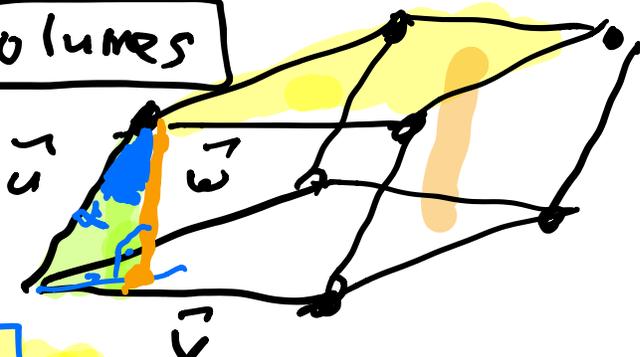
$\vec{v} \cdot \vec{w}$ scalar
 $\vec{v} \times \vec{w}$ vector

~~$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} * \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 20 \end{bmatrix}$~~

h connects para

③ Volumes

Parallelepiped



$$V \cdot W = |V| |W| \cos \alpha$$

$$\text{Volume} = \text{base area} \cdot \text{height}$$

$$= |\vec{v} \times \vec{w}| |\vec{u}| \cos \alpha$$

$$= |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$


$$\text{Volume of Parallelepiped} = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

$$\det \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) \quad \text{triple scalar product}$$

$$\vec{v} \cdot (\vec{w} \times \vec{u})$$

$$= \vec{w} \cdot (\vec{u} \times \vec{v})$$

$$= -\vec{u} \cdot (\vec{w} \times \vec{v})$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

right hand rule

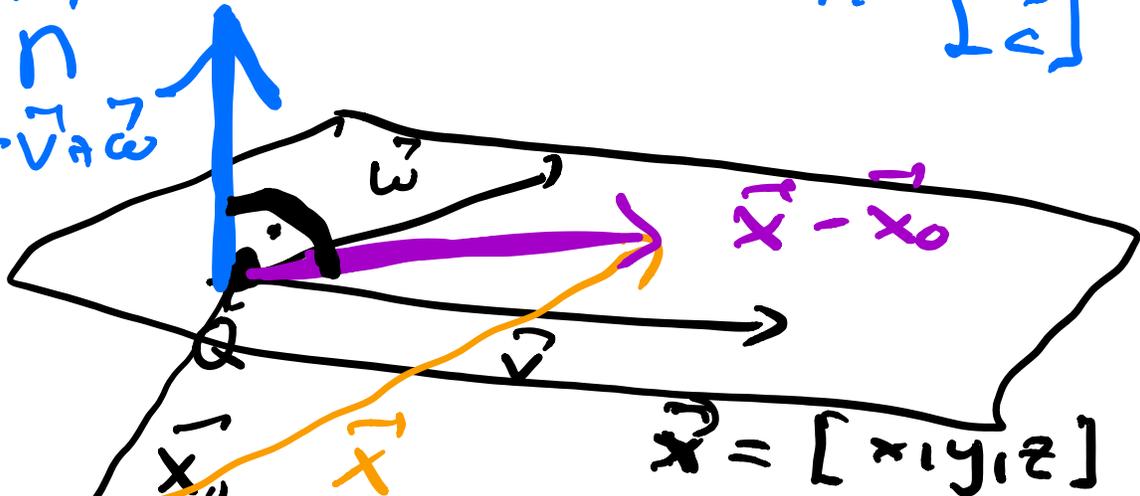
$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

④

Planes

$$n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2$$



$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$[a, b, c] \cdot [x, y, z] = d$$

$$ax + by + cz = d$$

equation of a plane

$$A = (1, 2, 3)$$

$$B = (3, 7, 0)$$

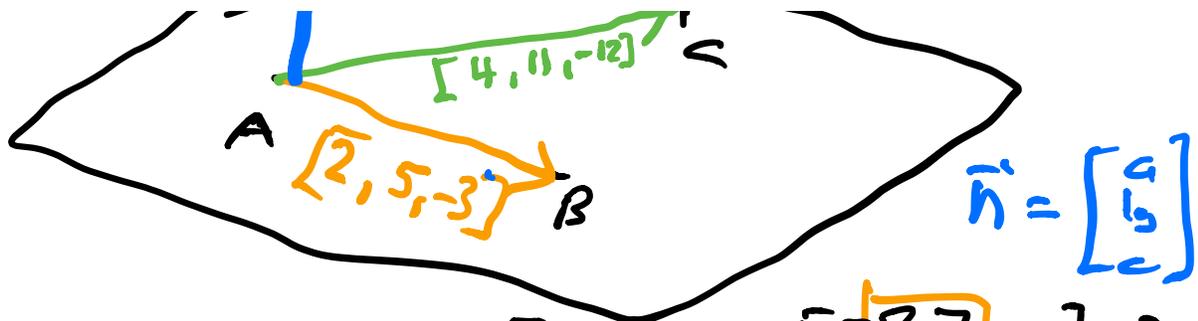
$$C = (5, 13, -9)$$

\vec{n}



Task:

Find the equation of the plane through A, B, C



$$\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 11 \\ -12 \end{bmatrix} = \begin{bmatrix} -27 \\ 12 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\begin{matrix} 2 & & 4 \\ 5 & \times & 11 \\ -3 & \times & -12 \\ 2 & & 4 \\ 5 & \times & 11 \end{matrix}$

$$\boxed{-27x + 12y + 2z = 3}$$

plug in a point \nearrow

$$\boxed{x^2 + y^2 + z^2 = 1}$$

$$3x + 4y + 2z = 12$$

plane



$$x^2 + y^2 = 1$$

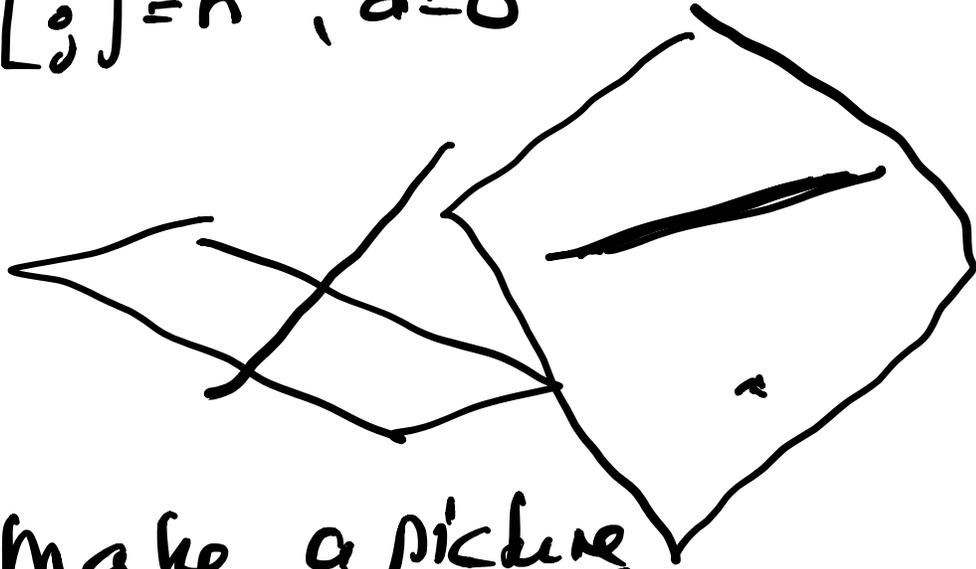
in space

$$x = 0$$



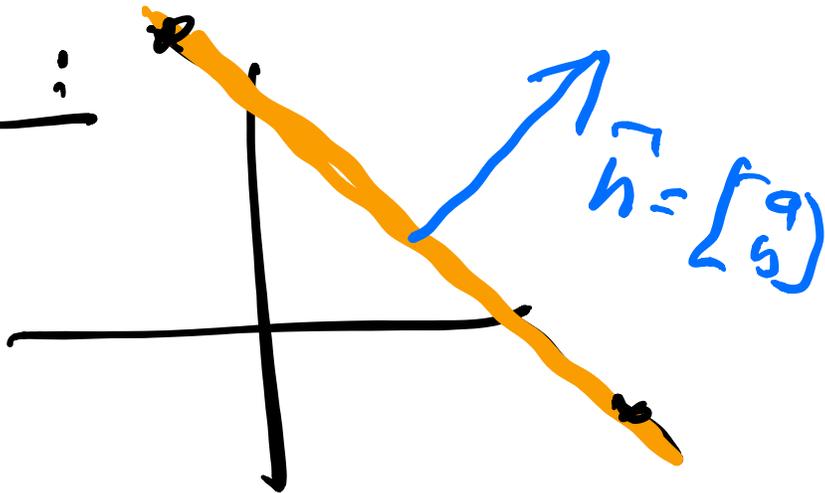
plane

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{n}, \quad d = 0$$



make a picture

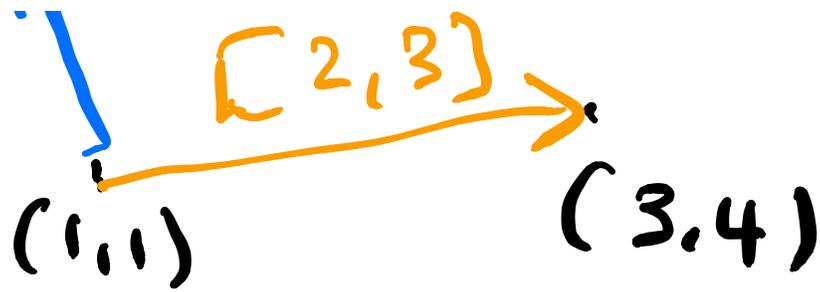
ZD :



$$ax + by = d$$

~~$$y = mx + b$$~~

~~↑ slope ↑ intercept~~



$$\vec{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \underline{\text{normal}}$$

$$\boxed{-3x + 2y = -1}$$

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ is } \perp \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$$

$x=0$
 $ax + by = d$

