$\boxed{\text{Unit 3}}$

1) Cross product

$$\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

$$\begin{matrix} 3 & & 2 \\ 4 & & 1 \end{matrix}$$

$$\boxed{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{matrix} 1 & & 2 \\ 1 & & 3 \end{matrix}$$

2) Properties

(i) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

anticommutative

(ii) $\vec{v}\times\vec{w} \perp \vec{v}$

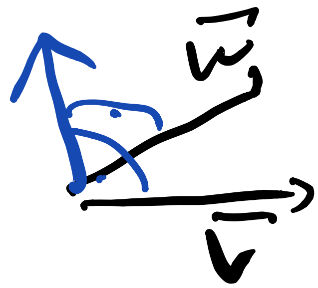
$\vec{v}\times\vec{w} \perp \vec{w}$

$\binom{n}{2} = \frac{n(n-1)}{2}$

$\binom{2}{2} = 1$ $\quad \binom{3}{2} = 3$

$\begin{bmatrix}2\\1\end{bmatrix}\times\begin{bmatrix}1\\1\end{bmatrix} = 1$

det



$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\cdot\begin{bmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{bmatrix}$$

$$= \boxed{v_1v_2w_3} - \boxed{v_1v_3w_2} + \boxed{v_2v_3w_1} - \boxed{v_2v_1w_3} + \boxed{v_3v_1w_2} - \boxed{v_3v_2w_1} = 0$$

(candy-crush)

do the same for $\vec{w}$

(iii) $\boxed{|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\alpha}$

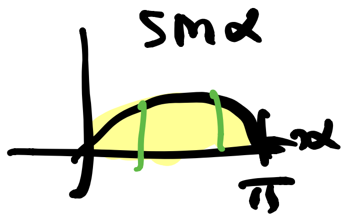
Cauchy-Binet: $|v|^2|w|^2 = |v\cdot w|^2 + |v\times w|^2$

→ in comp

$|\vec{v}\times\vec{w}|^2 = |v|^2|w|^2 - |\vec{v}|^2|\vec{w}|^2\cos^2\alpha$

because $|\vec{v}\cdot\vec{w}|^2 = |v|^2|w|^2\cos^2\alpha$

$$|\vec v \times \vec w|^2 = |\vec v|^2 |w|^2 (1-\cos^2\alpha)$$
$$= |\vec v|^2 |\vec w|^2 \sin^2\alpha$$



$$|\vec v \times \vec w| = |\vec v|\,|\vec w| \sin\alpha$$

QED

(iv) $|\vec v \times \vec w|$ is the area of the parallelogram spanned by v, w

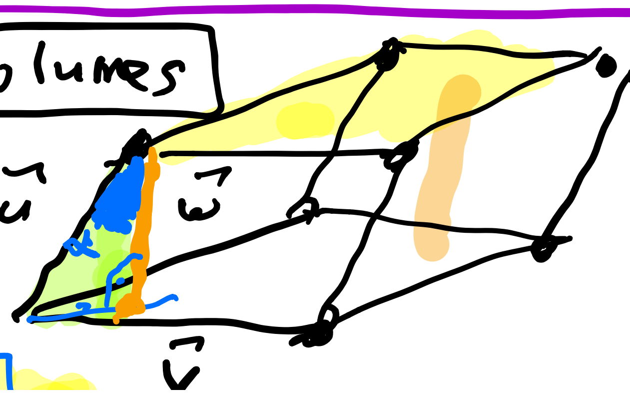
$$A = |v|\,\boxed{\text{height}}$$
$$= |v|\,|w| \sin\alpha$$
$$= |\vec v \times \vec w|$$

$\vec v \cdot \vec w$ scalar

$\vec v \times \vec w$ vector

$$\cancel{\begin{bmatrix}2\\3\\4\end{bmatrix} * \begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}6\\12\\20\end{bmatrix}}$$ ~~kronecker prod~~

③ $\boxed{\text{Volumes}}$



Parallelepiped

$\boxed{\vec v\cdot\vec w = |\vec v|\,|\vec w|\cos\alpha}$

Volume = base area · height

$$= |\vec v\times\vec w|\,|\vec u|\,|\cos\alpha|$$

$$= |(\vec v\times\vec w)\cdot\vec u|$$

$$\boxed{\text{Volume of Parallelepiped} = |\vec u\cdot(\vec v\times\vec w)|}$$

$$\det\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_1 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

$\vec u\cdot(\vec v\times\vec w)$ triple scalar product

$\parallel$

$$\vec v\cdot(\vec w\times\vec u)$$

$$= \vec w\cdot(\vec u\times\vec v)$$

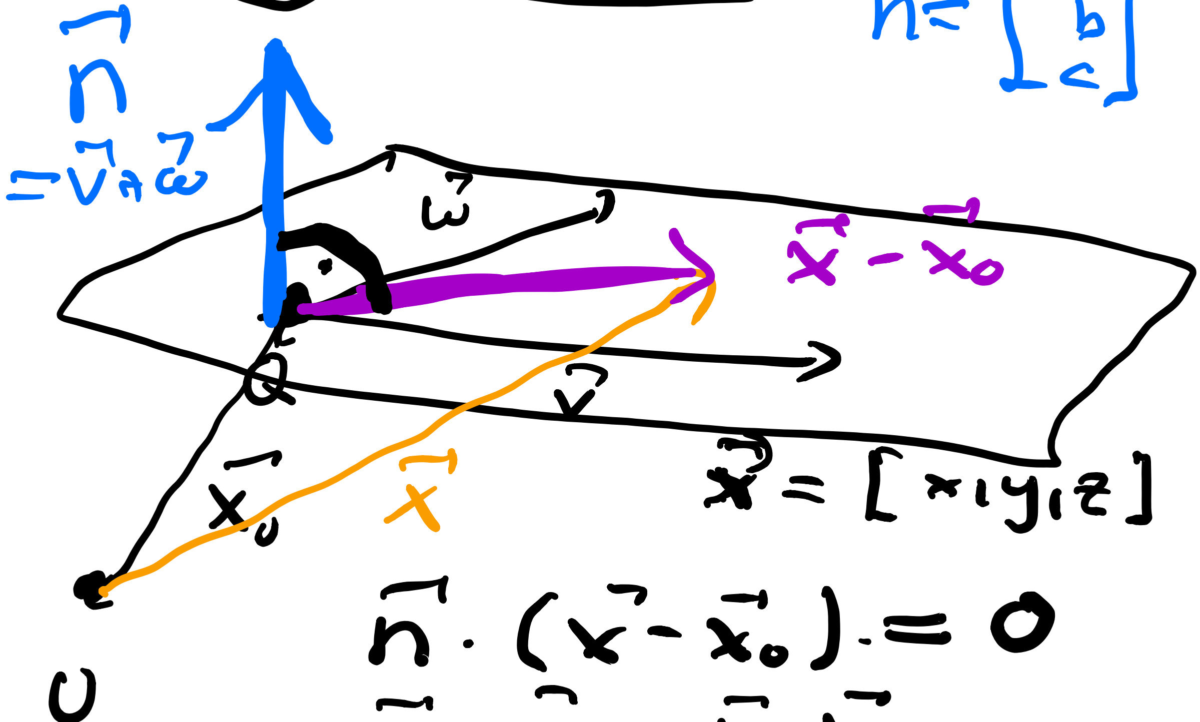
$$\boxed{\begin{aligned} &= -\vec u\cdot(\vec w\times\vec v) \\ &= \vec u\cdot(\vec v\times\vec w) \end{aligned}}$$

right hand rule

$$\vec v\times\vec w = -\vec w\times\vec v$$

④ Planes

$n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$



$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$[a, b, c] \cdot [x, y, z] = d$$

$$\boxed{ax + by + cz = d}$$

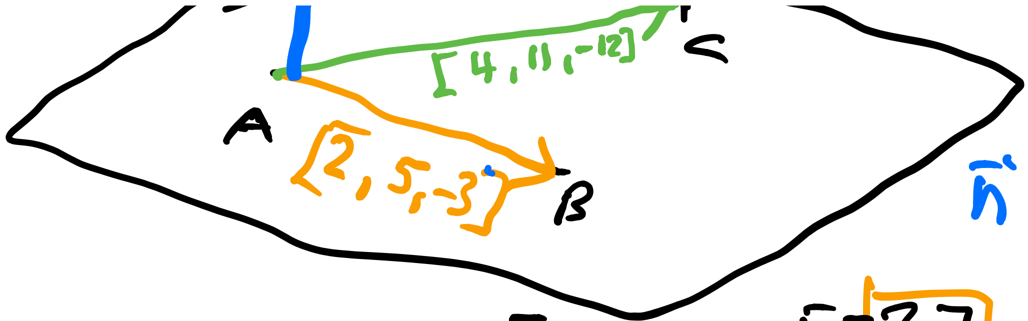
equation of a plane

$A = (1, 2, 3)$

$B = (3, 7, 0)$

$C = (5, 13, -9)$

Task: Find the equation of the plane through A, B, C



$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

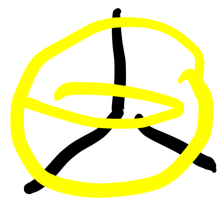
$$\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 11 \\ -12 \end{bmatrix} = \begin{bmatrix} -27 \\ 12 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{matrix} 2 & 4 \\ 5 & 11 \end{matrix}$$

$$\boxed{-27x + 12y + 2z = 3}$$

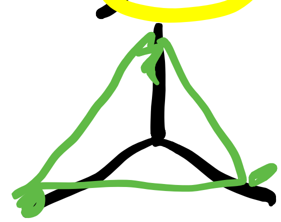
plug in a point

$$\boxed{x^2 + y^2 + z^2 = 1}$$



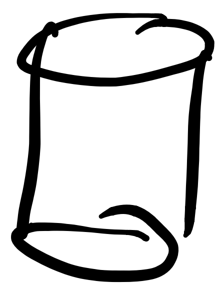
$$3x + 4y + 2z = 12$$

plane

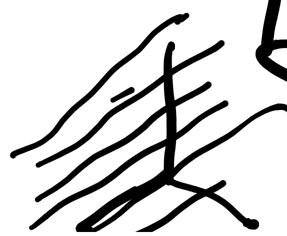


$x^2 + y^2 = 1$

in space



$x = 0$

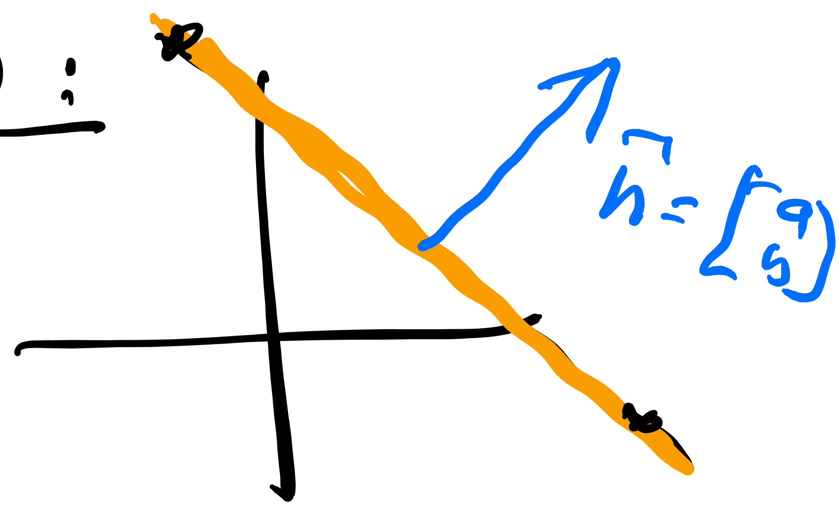


plane

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{n}$, $d = 0$

<u>make a picture</u>

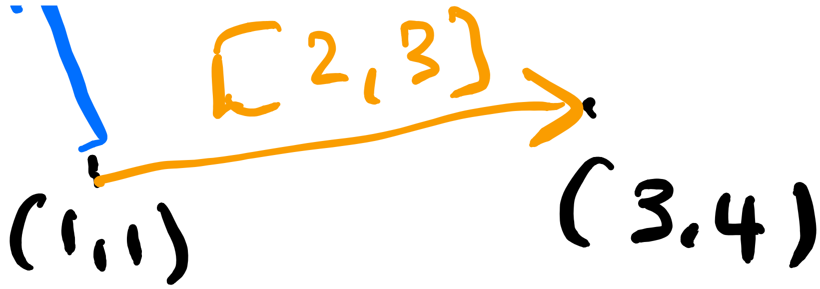
<u>2D:</u>



$$ax + by = d$$

$$\cancel{y = mx + b}$$

slope intercept

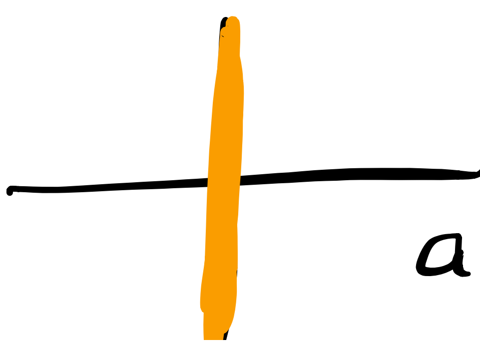


$\vec{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ <u>turned</u>

$$\boxed{-3x + 2y = -1}$$

$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is $\perp$ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$$



$x = 0$

$ax + by = d$