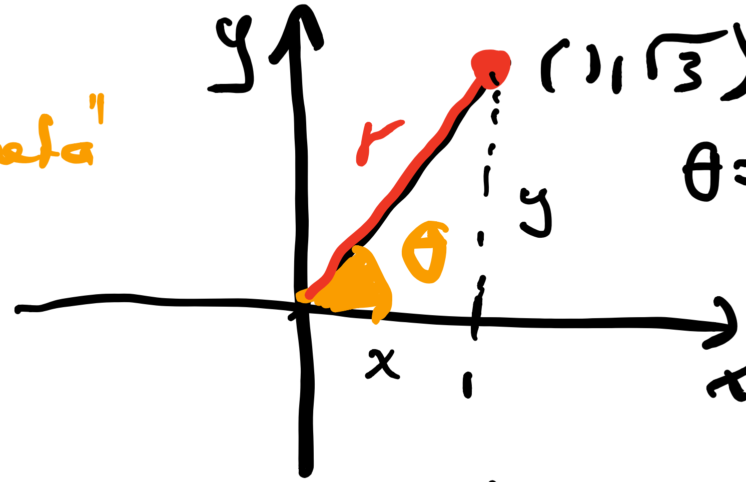
Unit 6

Parametric Surfaces

① Polar coordinates

θ "theta"



$(1, \sqrt{3}) = P$

$\theta = \frac{\pi}{3} = 60^\circ$

$r = 2$

(x, y) Cartesian coordinates

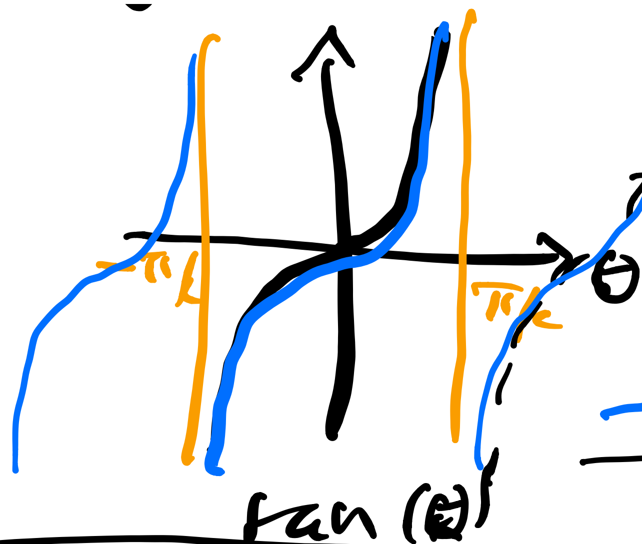
$(r, \theta) = (2, \frac{\pi}{3})$ Polar coordinates

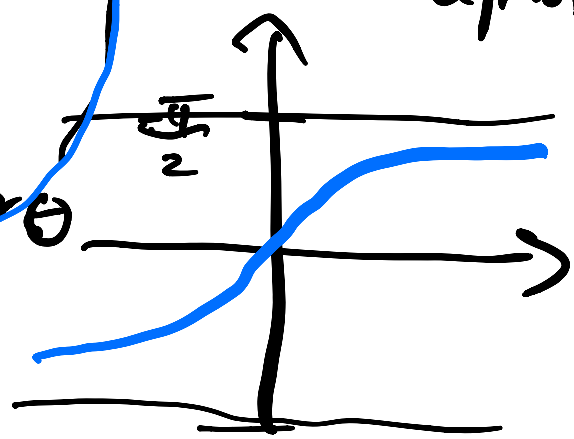
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

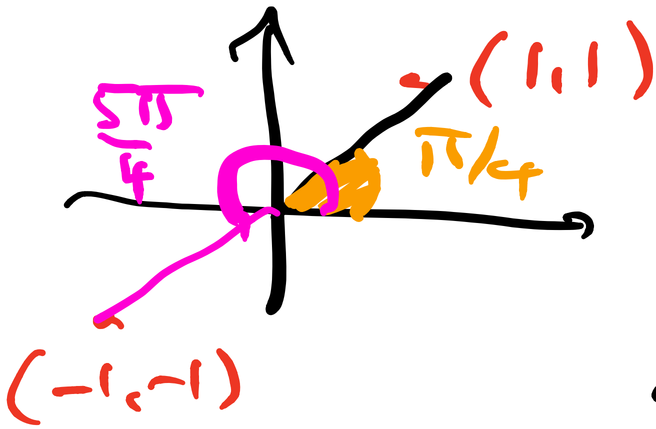
$r \geq 0$

$r = \sqrt{x^2+y^2} \geq 0$

$\theta = \arctan(y/x) \leftarrow$ only defined up to π

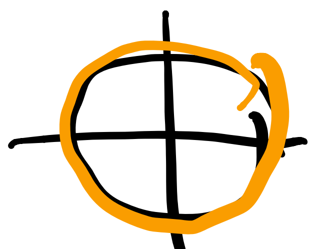






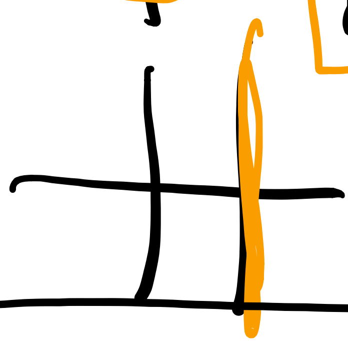
$\arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\arctan\left(\frac{-1}{-1}\right) = \frac{\pi}{4}$



$\boxed{r = 1}$ circle

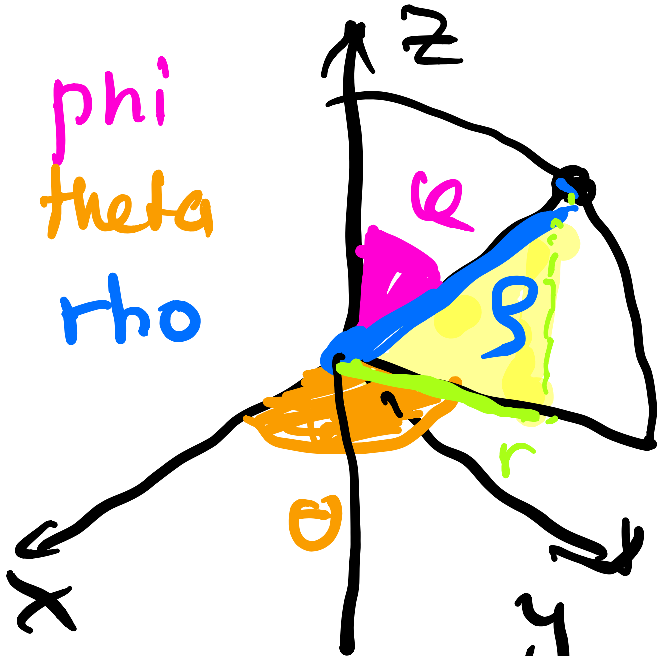
$\boxed{r\cos\theta = 1}$



② Spherical coordinates

$\boxed{\text{Euler}}$

φ phi
θ theta
ρ rho

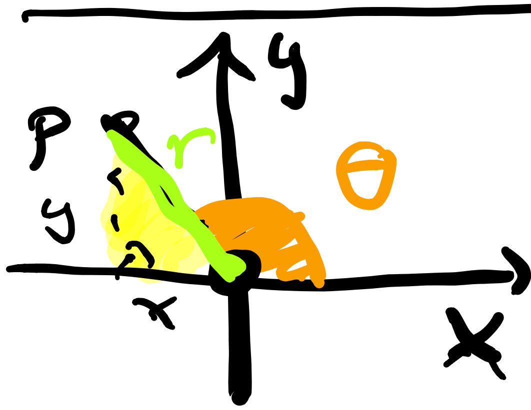


$P = (x, y, z)$

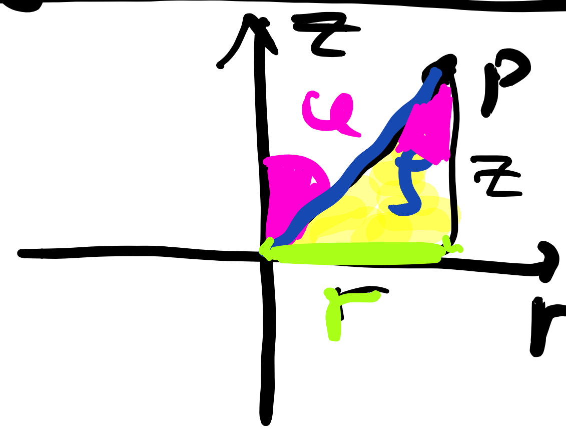
$0 \le \theta < 2\pi$

$0 \le \varphi \le \pi$

$\rho \ge 0$



Top view



Side view

r distance to z axis
not part of spherical but it helps.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \rho\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}}$$

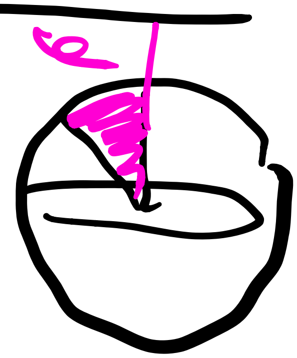
$$\boxed{\begin{aligned} x &= \overbrace{\boxed{\rho\sin\varphi}}^{r}\cos\theta \\ y &= \boxed{\rho\sin\varphi}\sin\theta \\ z &= \rho\cos\varphi \end{aligned}}$$

Problems

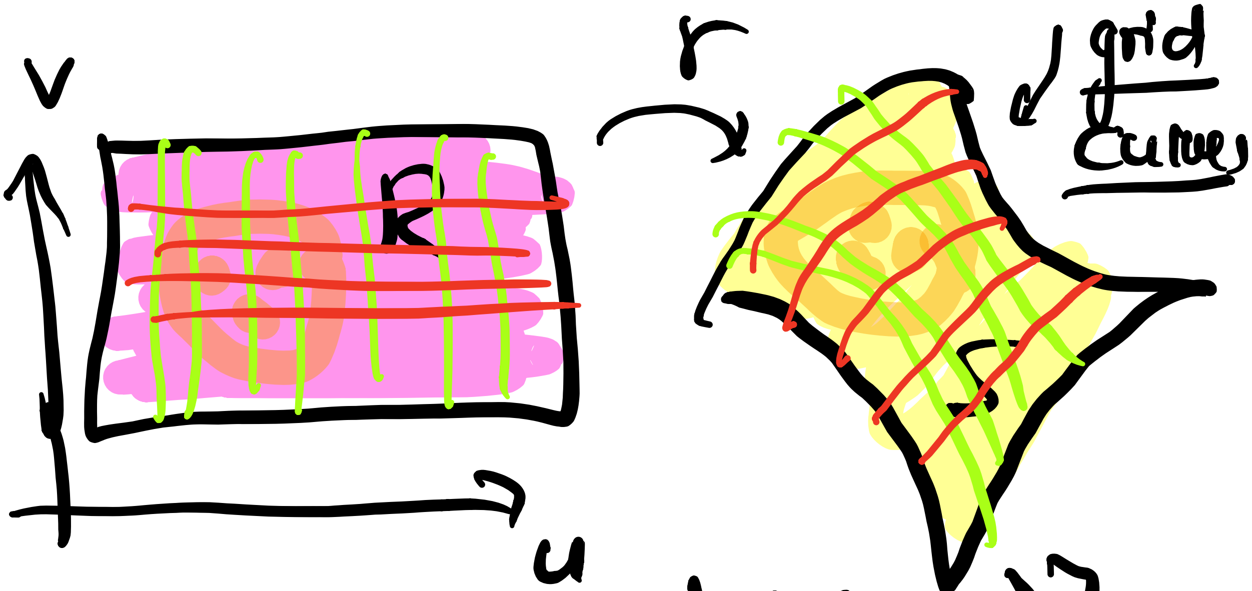
a) $\boxed{\rho = 1}$ Sphere

b) $\boxed{\rho = \dfrac{1}{\cos\varphi}}$ $z = 1$ plane

$\underbrace{\rho\cos\varphi}_{z} = 1$



③ Parametrization



$$\bar{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

uv-map
parametrization

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④ Examples: sphere

$0 \qquad 2\pi \quad \theta$

$$\boxed{\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\sin\varphi \cos\theta\\ \sin\varphi \sin\theta\\ \cos\varphi\end{bmatrix}}$$

parametrization of a sphere of radius 1

Prob. Parametrize: $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

$$\vec{r}(\theta,\varphi) = \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\sin\varphi \cos\theta\\ 3\sin\varphi \sin\theta\\ 4\cos\varphi\end{bmatrix}$$

$$\frac{(x-1)^2}{4} + \frac{(y+3)^2}{9} + \frac{z^2}{16} = 1$$

$$\vec{r}(\theta,\varphi) = \begin{bmatrix}1 + 2\sin\varphi \cos\theta\\ -3 + 3\sin\varphi \sin\theta\\ 0 + 4\cos\varphi\end{bmatrix}$$

⑤ $\boxed{\text{Graphs}}$

$$z = f(x,y) = x^2+y^2$$

How do we parametrize this? This is hard because it is easy!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix}$$

use x,y as variables
$\downarrow$
$u \quad v$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ u^2+v^2 \end{bmatrix} = \vec{r}(u,v)$$

Parameterization

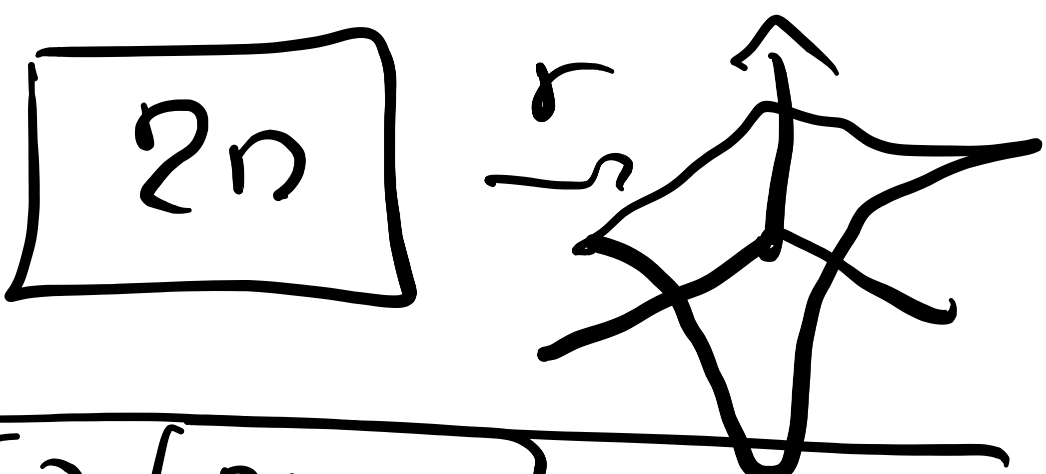
$\boxed{\mathbb{R}^m \to \mathbb{R}^n}$

$\mathbb{R} \to \mathbb{R}^3$ curves

$\mathbb{R}^2 \to \mathbb{R}^3$ surface

$\mathbb{R}^2 \to \mathbb{R}$ function

$\mathbb{R}^3 \to \mathbb{R}$ func.



(6) Planes review

$$\vec{r}(s,t) = \begin{bmatrix} 2 + s + 3t \\ 4 + 2s - t \\ \ldots \end{bmatrix}$$

$- 5 + 5 + 4 \cdot 5]$

parametrization of a plane.

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$$

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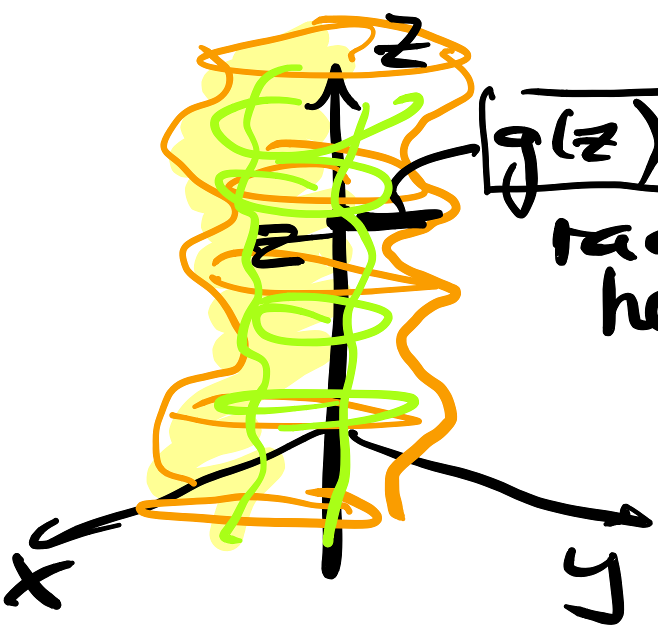
7 Surfaces of revolution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} g(z) \cos\theta \\ g(z) \sin\theta \\ z \end{bmatrix}$$



$\boxed{g(z)} = r$
radius at height z

Example:

a) Parametrize: $x^2 + y^2 = z^2$

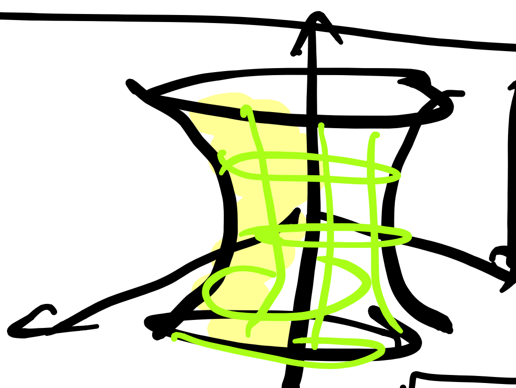
$\underbrace{x^2+y^2}_{r^2}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z\cos\theta \\ z\sin\theta \\ z \end{bmatrix}$$

because: $r^2 = z^2$

$g(z) = z$

b)

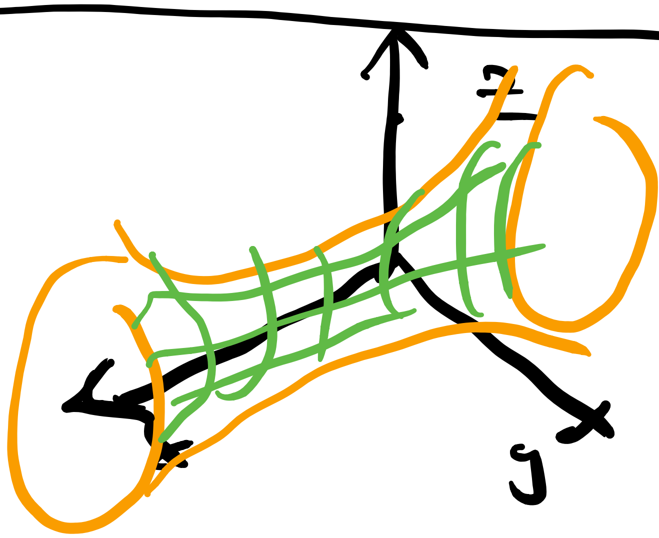


$$\boxed{x^2 + y^2 - z^2 = 1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{1+z^2}\cos\theta \\ \sqrt{1+z^2}\sin\theta \\ z \end{bmatrix}$$

c) $\boxed{y^2+z^2-x^2=1}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ \sqrt{1+x^2}\,\cos\theta \\ \sqrt{1+x^2}\,\sin\theta \end{bmatrix}$$

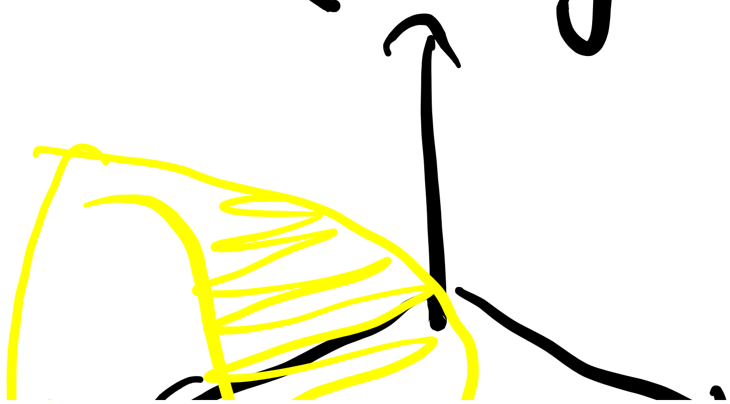


x, θ are the parameters

$r = y^2+z^2$

d)

$$x - y^2 - z^2 = 0$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y^2+z^2 \\ y \\ z \end{bmatrix}$$