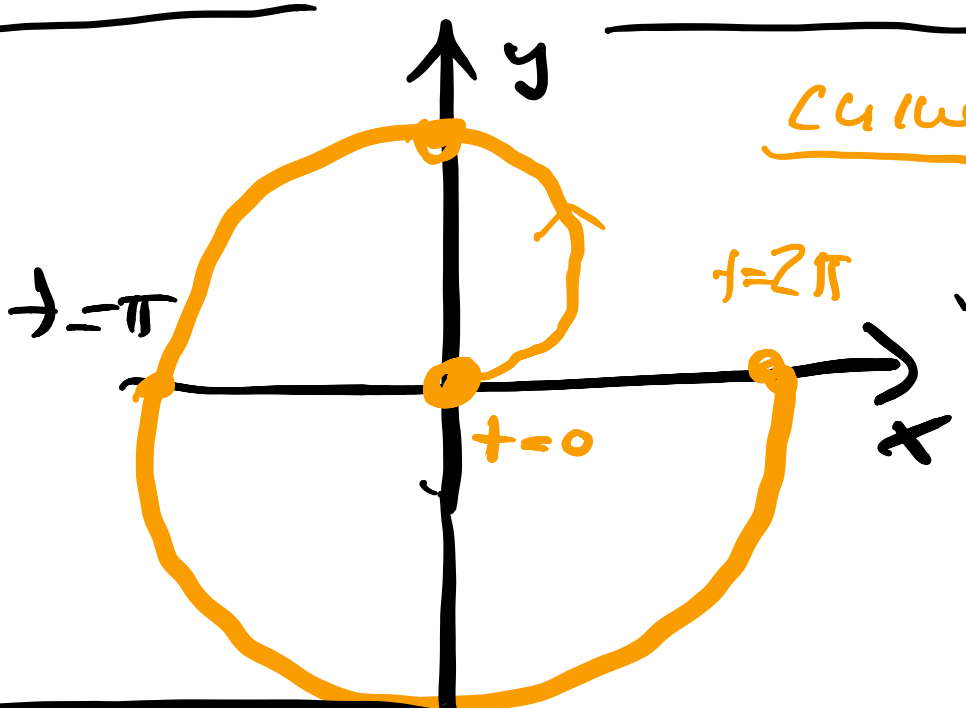
Unit 7

7 Curves

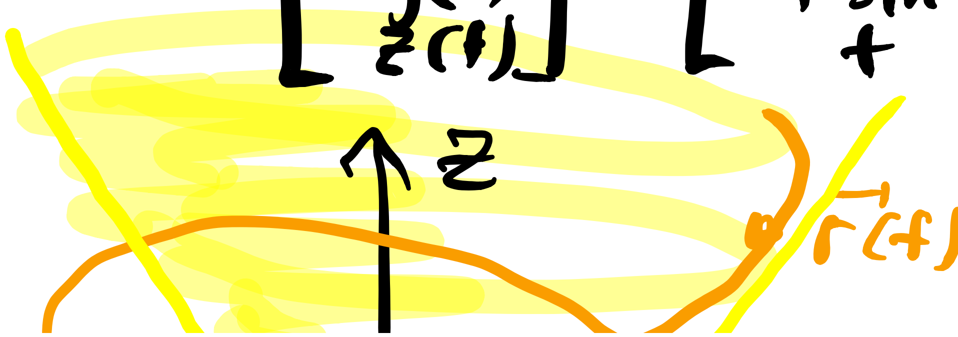
1) Parametrization

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}$$

$$0 \le t \le 2\pi \quad \text{time}$$



$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \\ t \end{bmatrix}$$



$$\boxed{x^2+y^2=z^2}$$ ✓

Find relations

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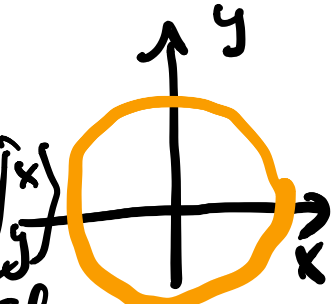
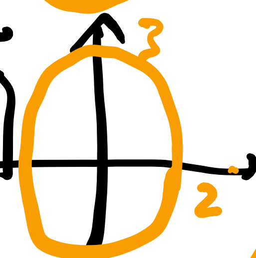
2) **Examples**

a) $\vec{r}(t)=\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$



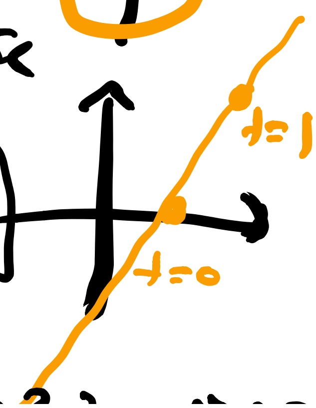
$x^2+y^2=1$ circle

b) $\vec{r}(t)=\begin{bmatrix}2\cos t\\ 3\sin t\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$



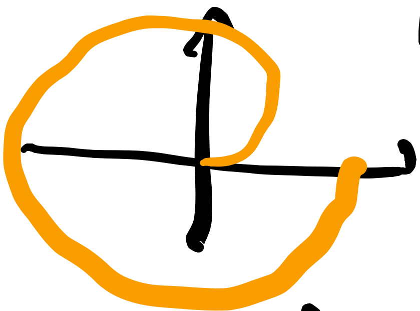
$$\boxed{\frac{x^2}{4}+\frac{y^2}{9}=1}$$ ellipse

c) $\vec{r}(t)=\begin{bmatrix}2t\\ 1+t\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$



Solve $2y-2=x$

d) $r(t) = \begin{bmatrix} t^3 \cos t^3 \\ t^3 \sin t^3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$



$\begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Same curves but parametrized differently.

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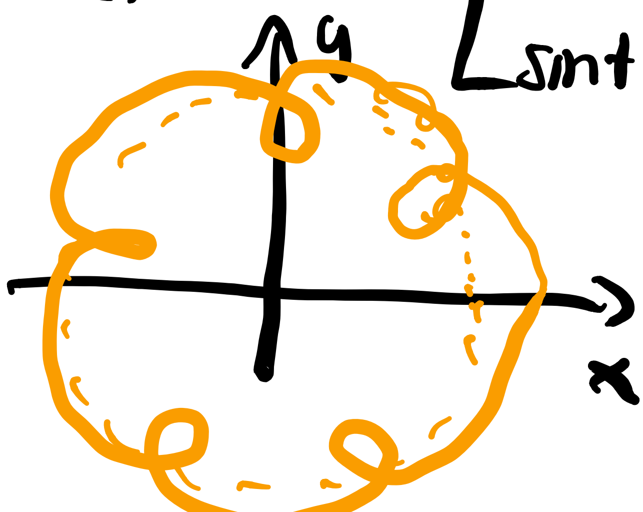
e) $\vec{r}(t) = \begin{bmatrix} \cos t + \dfrac{\cos 5t}{3} \\ \sin t + \dfrac{\sin 5t}{3} \end{bmatrix}$



epicycles

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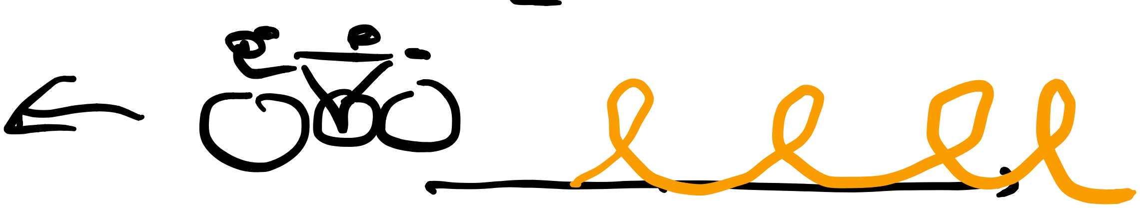
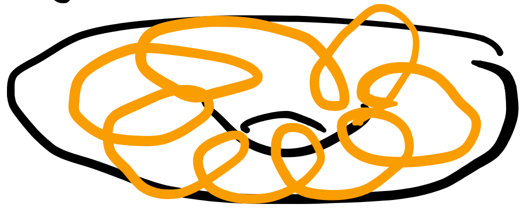
f) $\vec{r}(t) = \begin{bmatrix} t + \cos 5t \\ \sin 5t \end{bmatrix}$



g) $\vec{r}(t) = \begin{bmatrix} (3+\cos 10t)\cos 13t \\ (3+\cos 10t)\sin 13t \\ \sin 10t \end{bmatrix}$

what curve is there.



h) $r(t) = \begin{bmatrix} \cos 3t \sin 2t \\ \sin 3t \sin 2t \\ \cos 2t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$x^2+y^2+z^2 = 1$

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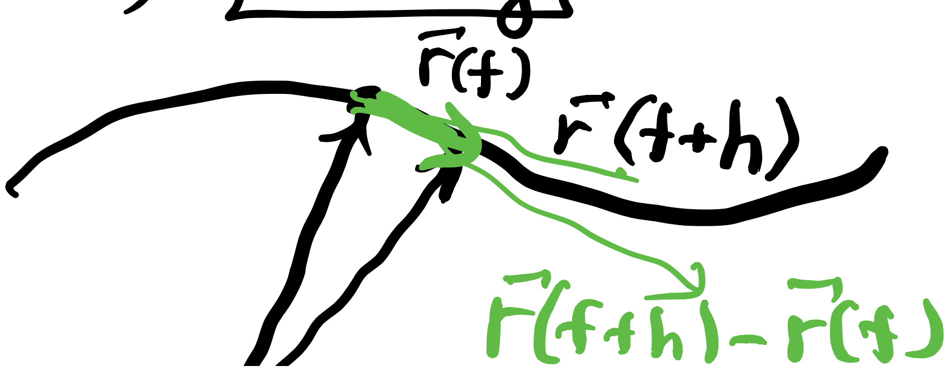
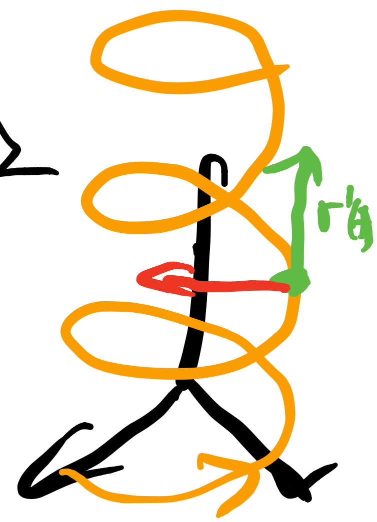
7) $\boxed{\text{Velocity}}$



$$\begin{bmatrix} \dfrac{x(t+h)-x(t)}{h} \\ \dfrac{y(t+h)-y(t)}{h} \\ \dfrac{z(t+h)-z(t)}{h} \end{bmatrix} \qquad \boxed{\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t) = \dot{\vec{r}}(t)}$$

velocity



$$\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$$

$$\boxed{\vec{r}\,'(t)} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$ velocity

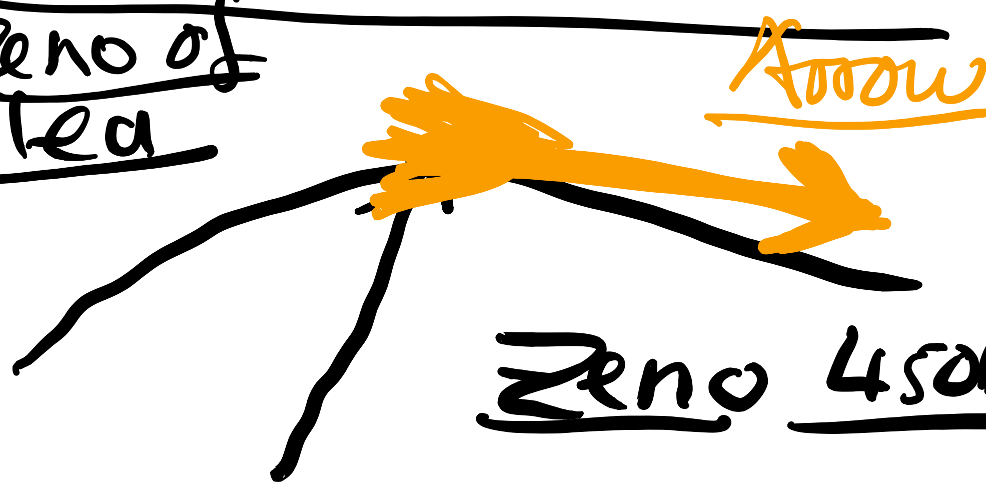
$$\boxed{\vec{r}\,''(t)} = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}$$ acceleration

Newtons law: $\boxed{m \cdot r''(t) = F}$

mass Force

$$\vec{r}'''(t) = \text{Jerk}$$
$$\vec{r}''''(t) = \text{Snap}$$
$$\vec{r}'''''(t) = \text{Crackle}.$$
$$\vec{r}''''''(t) = \text{pop}$$
$$\vec{r}'''''''(t) = \text{Hazard}$$

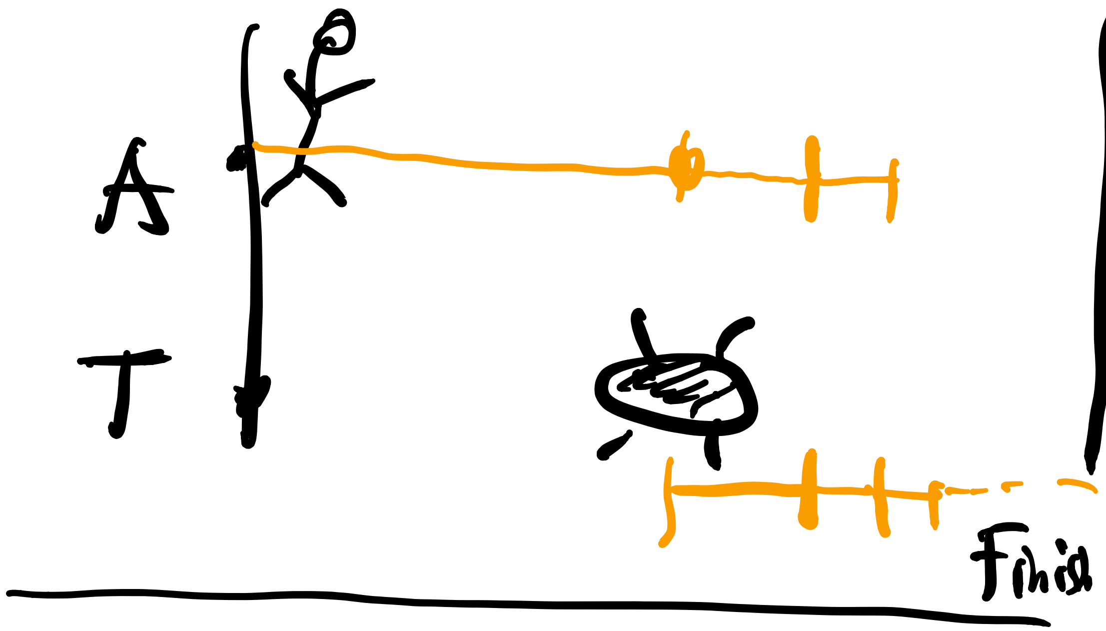
Zeno of Elea

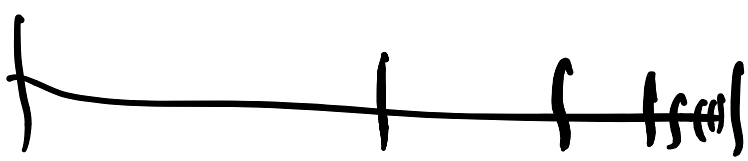


Zeno 450BC

Zeno paradox :

Moving arrow :





$f = 2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13$

$f' \quad 1 \quad 2 \quad 2 \quad 4 \quad 2$

velocity of the primes

$f' = 2$ means prime twin

4 $\boxed{\text{Integration}}$

a) $\vec{r}\,'(t) = \begin{bmatrix} t^2 \\ t^3 \\ 1+t \end{bmatrix}$

$\vec{r}(0) = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

where are we at $t=10$?

(Predict the future by knowing the derivative)

$$\vec{r}(t) = \begin{bmatrix} \frac{t^3}{3} + 5 \\ \frac{t^4}{4} + 6 \\ t + \frac{t^2}{2} + 7 \end{bmatrix}$$

b) $\begin{cases} \vec{r}''(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \quad \vec{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{r}'(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{r}(t) = \, ? \quad \vec{r}(100\pi) = \, ? \end{cases}$

$$\vec{r}'' = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\vec{r}'(t) = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{r}'(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{\vec{r}(t) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + \begin{bmatrix} 2t + 1 \\ t + 1 \end{bmatrix}}$$

$\vec{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{r}(100\pi) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 200\pi + 1 \\ 100\pi + 1 \end{bmatrix}$$

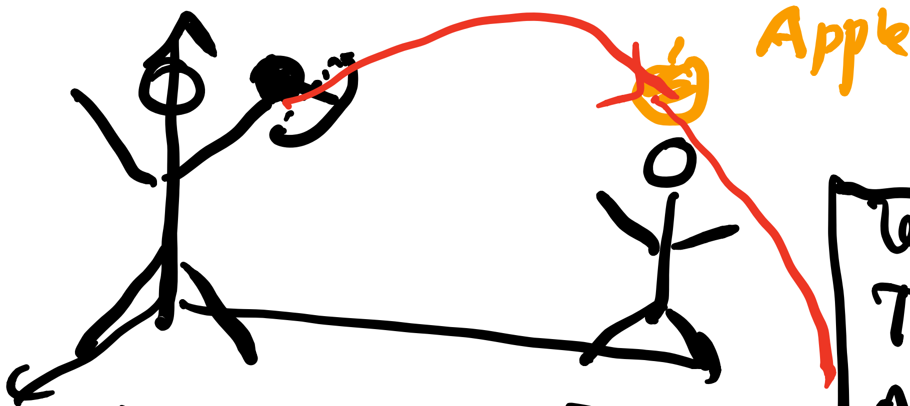
$$= \begin{bmatrix} 200\pi + 1 \\ 100\pi \end{bmatrix}$$

Application

$g = 9.96\ldots$

~ 10

$$\vec{r}''(t) = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$



Wilhelm Tell problem

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$\vec{r}'(0) = \begin{bmatrix} 200 \\ 1 \\ 3 \end{bmatrix}$$

where does $r(t)$ hit $z=0$?

$$\vec{r}''(t) = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

$$\vec{r}\,'(t) = \begin{bmatrix} 0 + 200 \\ 0 + 1 \\ -10t + 3 \end{bmatrix}$$

$$\vec{r}(t) = \begin{bmatrix} 200t \\ t \\ -5t^2 + 3t + 6 \end{bmatrix}$$

$$-5t^2 + 3t + 6 = 0$$

$$ax^2 + bx + c = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = -5$
$b = 3$
$c = 6$