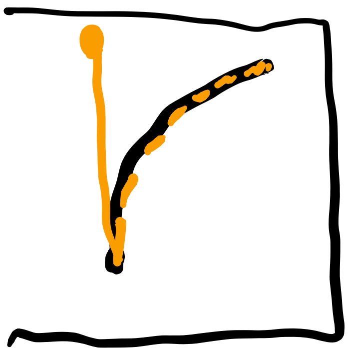
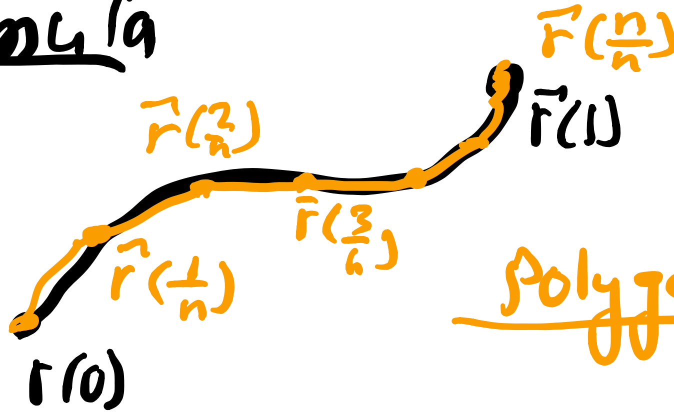
Unit 8 / Arc length and curvature

1) Formula





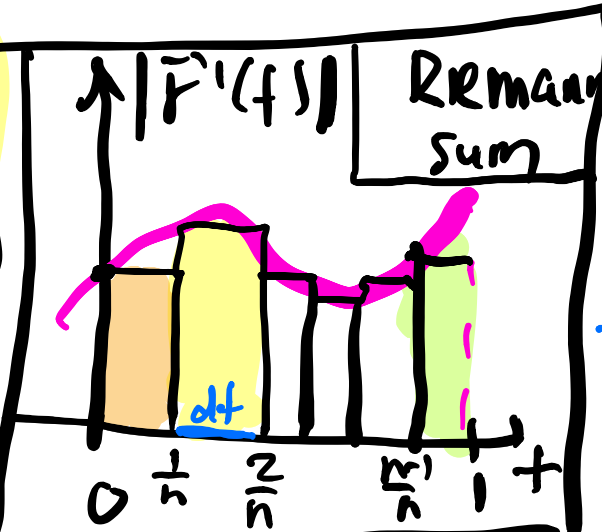
$$L_n = \left|\vec r\left(\tfrac{1}{n}\right) - \vec r\left(\tfrac{0}{n}\right)\right| + \left|\vec r\left(\tfrac{2}{n}\right) - \vec r\left(\tfrac{1}{n}\right)\right|$$

length of polygon

$$\dots + \left|\vec r\left(\tfrac{n}{n}\right) - \vec r\left(\tfrac{n-1}{n}\right)\right|$$

$$\vec r\left(\tfrac{k+1}{n}\right) - \vec r\left(\tfrac{k}{n}\right) \approx \frac{1}{n}\, \vec r\,'\left(\tfrac{k}{n}\right)$$

Difference quotient



$\frac{1}{n}$ dt

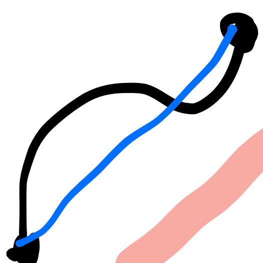
$$L_n = \frac{1}{n}\left[\left|\vec r\,'\left(\tfrac{0}{n}\right)\right| + \left|\vec r\,'\left(\tfrac{1}{n}\right)\right| + \left|\vec r\,'\left(\tfrac{n-1}{n}\right)\right|\right]$$

$$\xrightarrow{n\to\infty} \quad \boxed{\int_a |\vec r\,'(t)|\,dt = L}$$

odometer

Question: why is FTC

$$L = \int_0^1 |\vec r\,'(t)|\,dt$$



$$= |\vec r(1) - \vec r(0)|$$

$=$ direct distance

false ?

It is true that

$$\boxed{\int_0^1 \vec r\,'(t)\,dt = \vec r(1) - \vec r(0)}$$

This is the FTC

② Examples

(a) arc length of a circle of radius R

$$\boxed{x^2+y^2=R^2}$$

$$\vec{r}(t)=\begin{bmatrix} R\cos t \\ R\sin t \end{bmatrix}$$

$$\vec{r}\,'(t)=\begin{bmatrix} -R\sin t \\ R\cos t \end{bmatrix}$$

speed

$$|\vec{r}\,'(t)|=\sqrt{R^2\sin^2 t+R^2\cos^2 t}=\boxed{R}$$

$$\int_0^{2\pi} R\,dt=\boxed{2\pi R}$$

Archimedes 300 BC

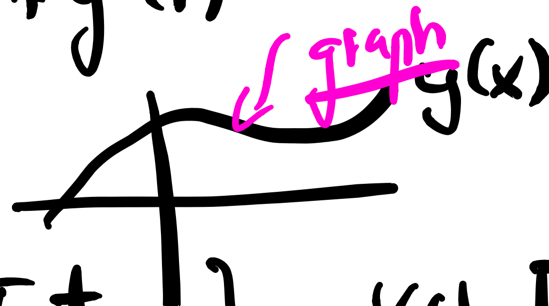
b) $\vec{r}(t) = \begin{bmatrix} \log t \\ \sqrt{2}\,t \\ \frac{t^2}{2} \end{bmatrix}$

$1 \le t \le 3$

Find the length.

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Single va



$$r(t) = \begin{bmatrix} t \\ y(t) \end{bmatrix} \qquad r'(t) = \begin{bmatrix} 1 \\ y'(t) \end{bmatrix}$$

$$|\vec{r}\,'(t)| = \sqrt{1 + y'(t)^2}$$

$$L = \int_a^b \sqrt{1 + y'(x)}\, dx$$

single variable point of view

$$\ln(x) = \log(x)$$

$$\vec{r}(t) = \begin{bmatrix} \log t \\ \sqrt{2}\,t \\ \frac{t^2}{2} \end{bmatrix}$$

$$1 \leq t \leq 3$$

$$\vec{r}\,'(t) = \begin{bmatrix} \frac{1}{t} \\ \sqrt{2} \\ t \end{bmatrix}$$

$$|\vec{r}\,'(t)| = \sqrt{\frac{1}{t^2} + 2 + t^2}$$

Simplify

$$= \frac{1}{t} + t$$

$(a+b)^2$
$= a^2 + 2ab + b^2$

opportunity.

$$\int_1^3 \frac{1}{t} + t \, dt = \log t + \frac{t^2}{2} \Big|_1^3$$

$$1$$

$$\left(\frac{1}{t}+t\right)^2 = \frac{1}{t^2} + 2\frac{1}{t}\cdot t + t^2$$

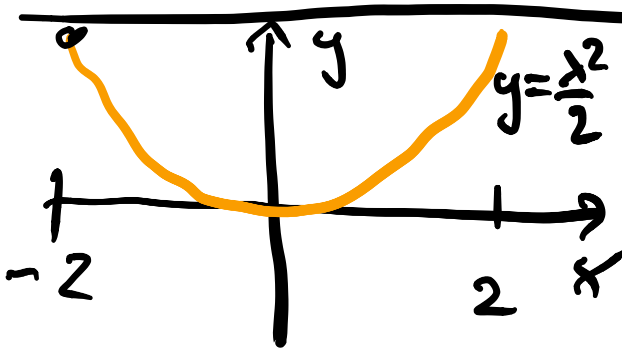
$$= \log 3 + \frac{9}{2}$$

$$- \cancel{\log(1)} - \frac{1}{2}$$

$$= \boxed{\log 3 + 4}$$

c)

dough



$\log_2(x)$

$\log_{10}(x)$

$\log(x)$

natural log

$$\vec{r}(t) = \begin{bmatrix} t \\ t^2/2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{r}'(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$|\vec{r}'(t)| = \sqrt{1+t^2}$$

$$\int_0^2 \sqrt{1+t^2}\,dt$$

Parametrize ~~graph~~ $y=f(x)$ $\qquad \vec{r}(t)=\begin{bmatrix} t \\ f(t) \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$

Remember! $\vec{r}(x,y)=\begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$x=\sin y \qquad \vec{r}(t)=\begin{bmatrix} \sin t \\ t \end{bmatrix}$

$$\int_{-2}^{2}\sqrt{1+t^2}\,1\,dt =$$

$\int u \; dv$ Integration by part $uv-\int v\,du$

$$I=\int_{-2}^{2}\sqrt{1+t^2}\cdot 1\,dt=\sqrt{1+t^2}\cdot t\Big|_{-2}^{2}-\int_{-2}^{2}\frac{2t\cdot t}{2\sqrt{1+t^2}}\,dt$$

$$=\sqrt{1+t^2}\cdot t\Big|_{-2}^{2}-\int_{-2}^{2}\frac{t^2}{\sqrt{1+t^2}}\,dt$$

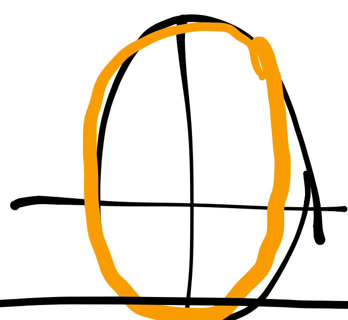
$$= \sqrt{1+t^2}\, t \Big|_{-2}^{2} - \int_{-2}^{2} \frac{t^2+1}{\sqrt{1+t^2}}\, dt + \int_{-2}^{2} \frac{1}{\sqrt{1+t^2}}\, dt$$

$$= \sqrt{1+t^2}\, t \Big|_{-2}^{2} - \underbrace{\int_{-2}^{2} \sqrt{1+t^2}\, dt}_{I} + \operatorname{arcsinh}(t)\Big|_{-2}^{2}$$

$$2I = 2\sqrt{1+4}\cdot 2 + \operatorname{arcsinh} 2$$

$$I = \boxed{2\sqrt{5} + \operatorname{arcsinh} 2}$$

d) $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 3\sin t \end{bmatrix}$



Elliptic integrals

e) $\vec{r}(t) = \begin{bmatrix} \cos 3t \\ \sin 3t \\ \cos 7t \\ \sin 7t \end{bmatrix} \leftarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

$0 \le t \le 2\pi$

Curve on the hypersphere $\mathbb{R}^4$

$$\boxed{x^2+y^2+z^2+w^2=2}$$

$$\tilde{r}'(t) = \begin{bmatrix} -\sin 3t \cdot 3 \\ \cos 3t \cdot 3 \\ -\sin 7t \; 7 \\ \cos 7t \; 7 \end{bmatrix}$$

$$\bar{r}'(t) = \sqrt{9+49}$$

$$= \sqrt{58}$$

($\cos^2(3t) + \sin^2(3t) = 1$

$\cos^2(7t) + \sin^2(7t) = 1$

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③ $\boxed{\text{Curvature}}$

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$$

$$\vec{r}'(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|r'(t)|}$$ unit tangent vector

κ = kappa = curvature

$$\boxed{\kappa(t) = \frac{|T'(t)|}{|\vec{r}'(t)|}}$$ number

k!

Theorem : $T\uparrow$

$$\boxed{\frac{|\vec{r}'(t) \times r''(t)|}{|\vec{r}'|^3} = \kappa(t)}$$

$$\vec{r}(t) = \begin{bmatrix} r\cos t \\ r\sin t \\ t \end{bmatrix}, \quad r'(t) = \begin{bmatrix} -r\sin t \\ r\cos t \\ 1 \end{bmatrix}$$

$$|\vec{r}\,'(t)| = \sqrt{1 + r^2\cos^2 t + r^2\sin^2 t} = \sqrt{2}$$

$$1+r^2$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\sqrt{2}} = \begin{bmatrix} -\sin t/\sqrt{2} \\ \cos t/\sqrt{2} \end{bmatrix}$$

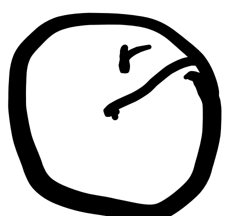
$$\vec{T}\,'(t) = \begin{bmatrix} -\cos t/\sqrt{2} \\ -\sin t/\sqrt{2} \end{bmatrix}$$

$$|\vec{T}\,'(t)| = 1/\sqrt{2}$$

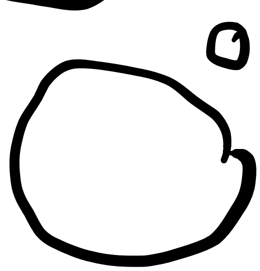
$$\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

$$\frac{r}{1+r^2}$$

Circle:

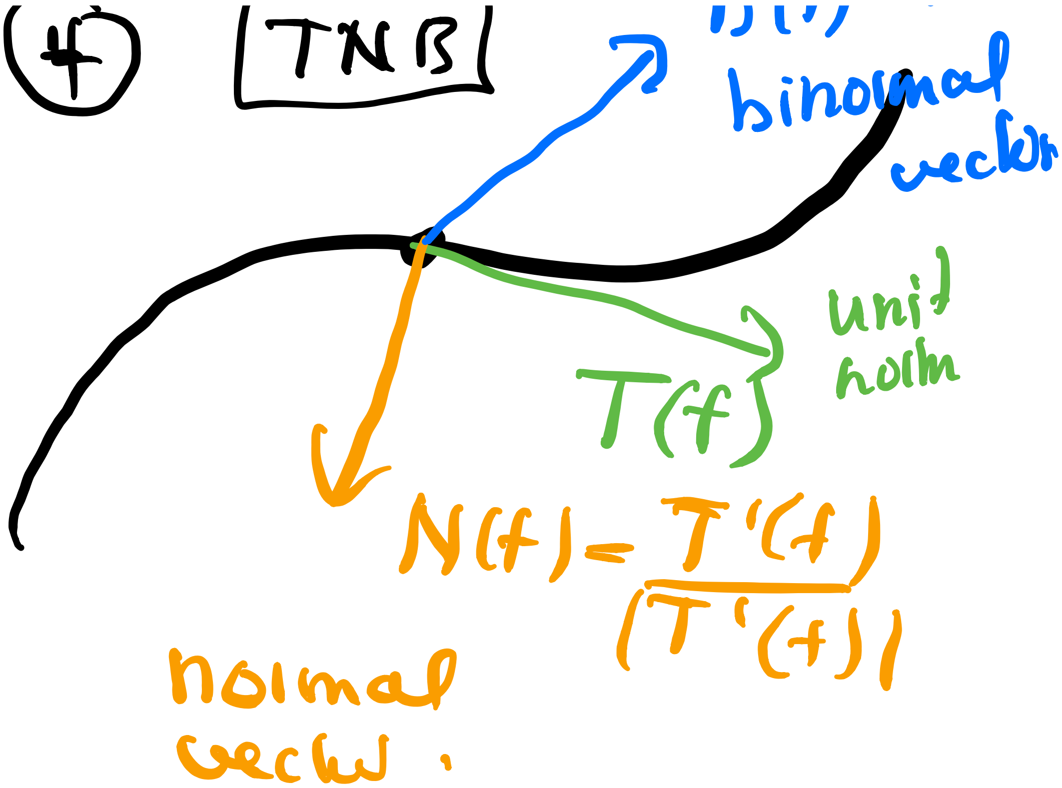


$$\boxed{\kappa = \frac{1}{r}}$$



$$B(t) = \vec{T} \times \vec{N}$$

(4) $\boxed{TNB}$



- unit tangent vector T direction in which you go
- Normal vector "left or right"
- Binormal vector "up"

<u>**Theorem:**</u> $\vec{T} \cdot \vec{N} = 0$

all three vectors are

perpendicular

Proof: $\vec{T} \cdot \vec{T} = 1$

$$\frac{d}{dt}(T \cdot T) = 0$$

$$= T' \cdot T + T \cdot T'$$

$$= \boxed{2T' \cdot T = 0}$$

$\bar{N} = \dfrac{T'}{|T'|}$ is perpend. to T

QED