# Unit 11 Chain rule

u substitution
compute area, inverse

Chain rule

## 1) 1D Case

$$\frac{d}{dx}\cos(e^x)$$

$$= -\sin(e^x)\,e^x$$

$$\frac{d}{dx}\log(\sin(\cos x))$$

$$= \frac{1}{\sin(\cos(x))}\cdot\cos(\cos(x))\cdot -\sin x$$

Schizophrenia

(Gollum $\leftrightarrow$ Smeagol)

variables function

$$\boxed{\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)}$$

2 Multivariable version

$f(x,y)$ function
$\vec{r}(t) = [x(t), y(t)]$
curve.

$f(\vec{r}(t)) = f(x(t), y(t))$
makes sense.

Think of $f$ as
<u>height</u>.
Think of $\vec{r}(t)$ as
a <u>curve</u> of a hike

$f(r(t))$ is the
height at time $t$.

$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

Chain rule in multi-variable

$$\nabla f = [f_x, f_y]$$

is the "derivative" is called gradient.

$$\vec{r}\,'(t) = [x'(t), y'(t)]$$

is the <u>velocity</u>.

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

<u>Remember</u>:

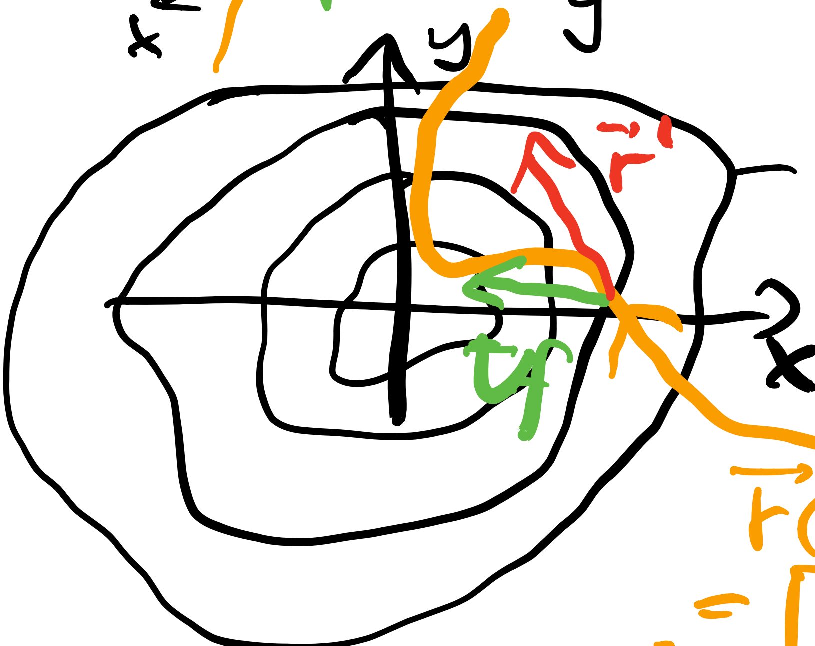
$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$


mountain

$f(x,y)$ height

$\vec{r}(t)$



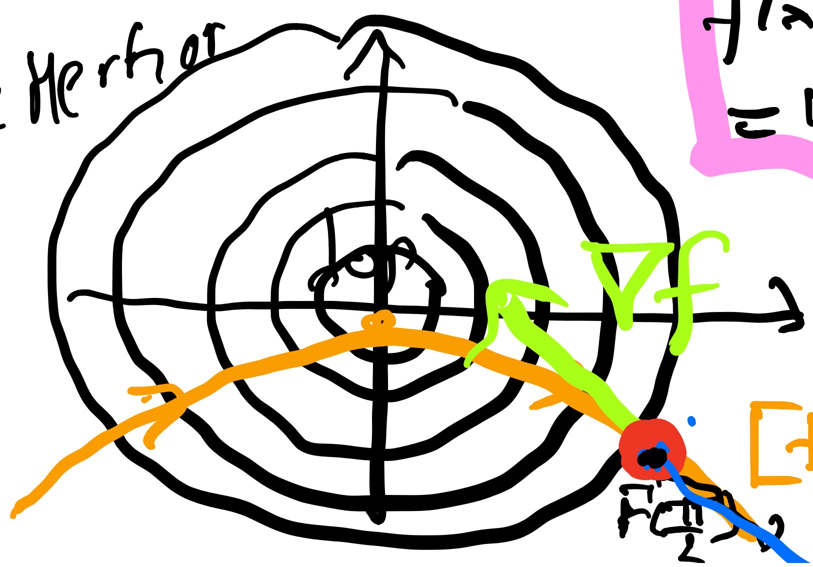

$f = \text{const}$

$\vec{r}(t) = [x(t), y(t)]$

3) Example

Matterhorn

$$\boxed{f(x,y) = 4500 - x^2 - y^2}$$




$\vec{r}(t)$

$[t, \cos t - 1]$

$\vec{r}\,'(\pi/2)$

Direct: $f(\vec{r}(t)) = 4500 - t^2 - (\cos t - 1)^2$

$$\frac{d}{dt} f(\vec{r}(t)) = -2t + 2(\cos t - 1)\sin t$$

$$\frac{d}{dt} f(\vec{r}(\tfrac{\pi}{2})) = -\pi + 2(0-1)\cdot 1 = \boxed{-\pi - 2}$$

We are climbing down.

---

Use chain rule:

point $\vec{r}(\frac{\pi}{2})$

$$\nabla f(x,y) = \begin{bmatrix} -2x \\ -2y \end{bmatrix}, \quad \nabla f\left(\boxed{\tfrac{\pi}{2}, -1}\right) = \boxed{\begin{bmatrix} -\pi \\ +2 \end{bmatrix}}$$

$$\vec{r}'(t) = [1, -\sin t]$$

$$\vec{r}'(\tfrac{\pi}{2}) = [1, -1]$$

Chain rule:

$$\frac{d}{dt} f(\vec{r}(t)) = \begin{bmatrix} -\pi \\ +2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boxed{-\pi - 2}$$

---

## (4) The ring

$$f(x,y) = xy$$

$$\vec{r}'(t) = [x(t), y(t)]$$

$$\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}, \quad r' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

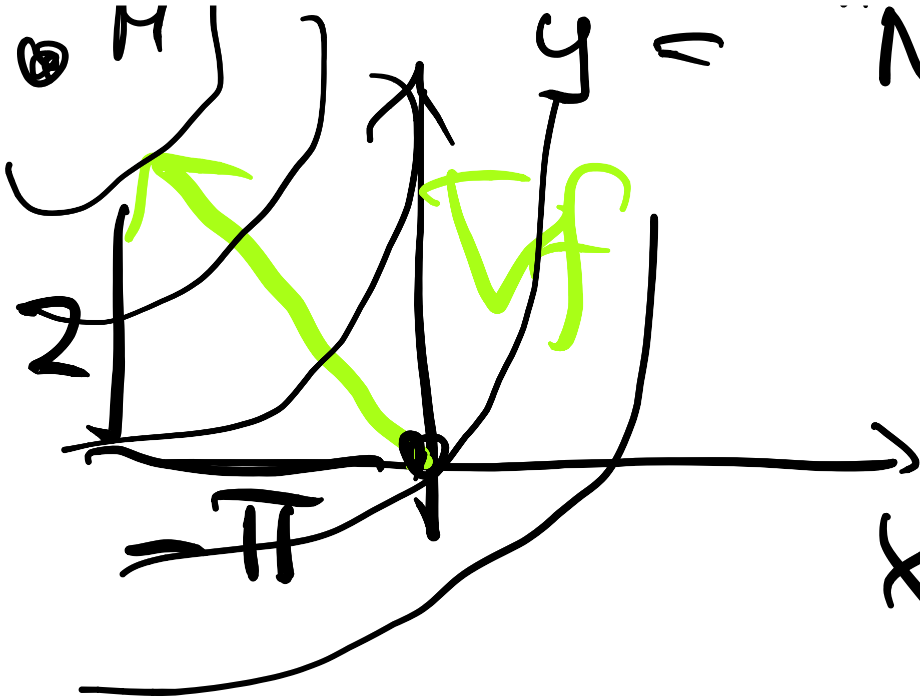
$$\frac{d}{dt}\big(x(t) \cdot y(t)\big) = y\,x'(t) + x\,y'(t)$$

This is the <u>product rule</u>

---

$$\left( \begin{array}{ccc} x(t) & \sim & f(x) \\ y(t) & \sim & g(x) \\ t & \sim & x \end{array} \right)$$

---




$y$ = "North"

$x = E$
east

"One ring to rule them all"