Unit 12 Gradient

① Gradient theorem

$\nabla f(x_0, y_0)$ is $\perp$ to the level curve $\boxed{f=c}$

Proof:

Let $\vec{r}(t)$ be a parametrization of $f=c$.

What is $\frac{d}{dt}\underbrace{f(\vec{r}(t))}_{=c}$?

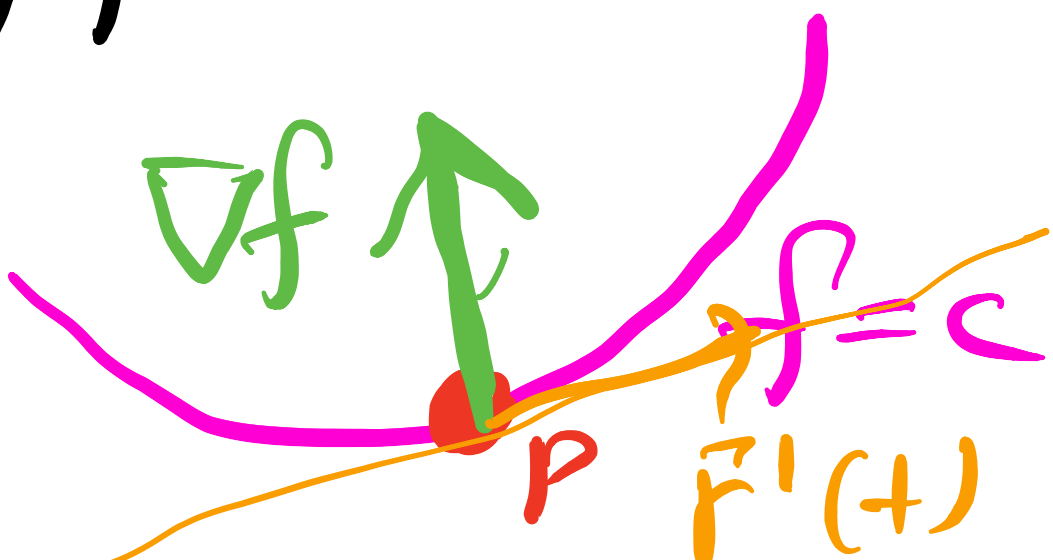
It is 0!

We do not change

height, when moving along $f=c$

Now use the chain rule!

$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$



$$f(x,y) = x^2 + y^2$$

$$x^2 + y^2 = 1$$

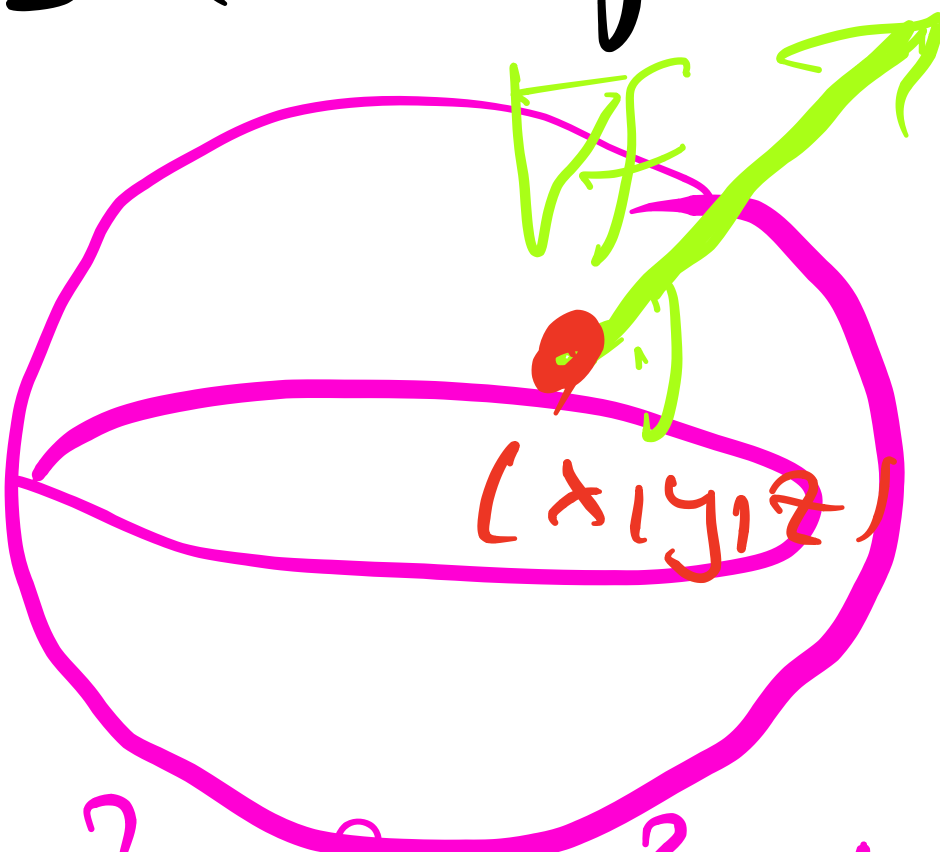
Circle

$$\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

In 3D

$f(x,y,z)=c$

is a surface



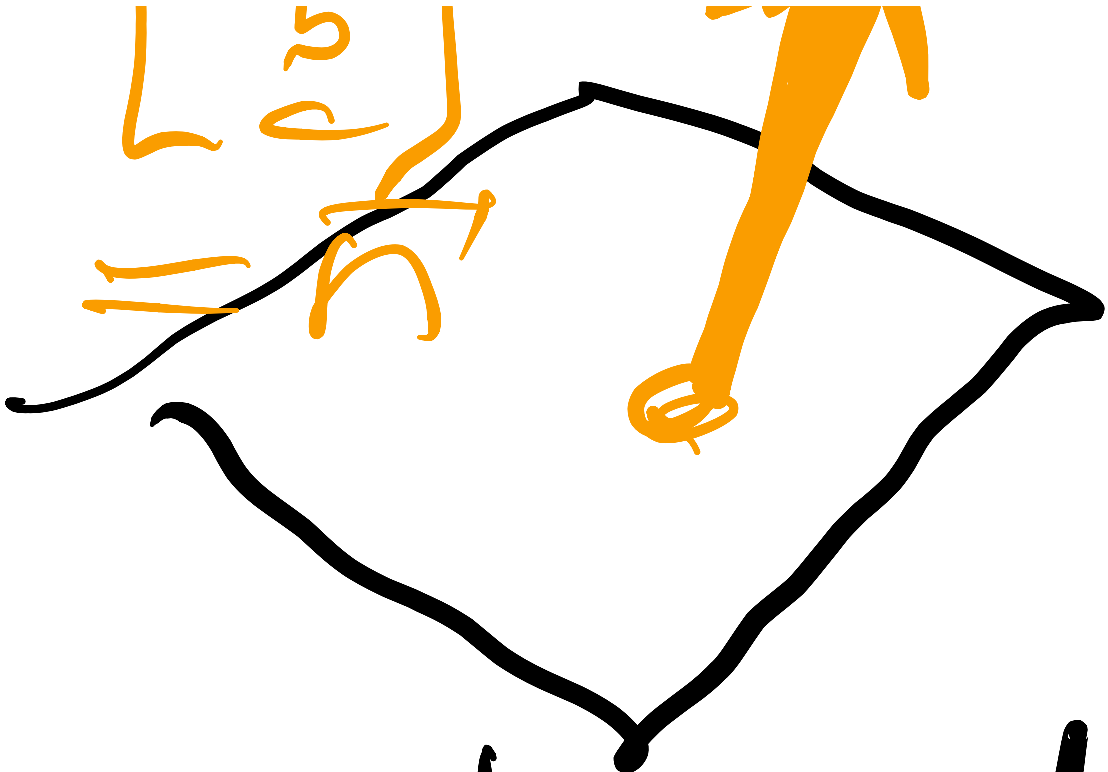
$f = x^2+y^2+z^2 = 1$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

Example:

$$f(x,y,z) = ax + by + cz$$

$$\nabla f = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$ax+by+cz=d$$

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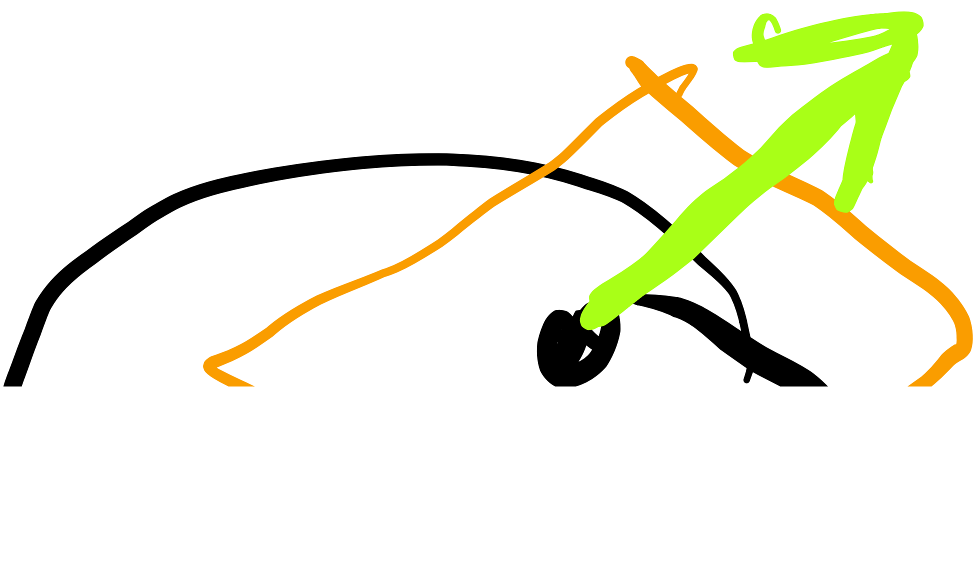
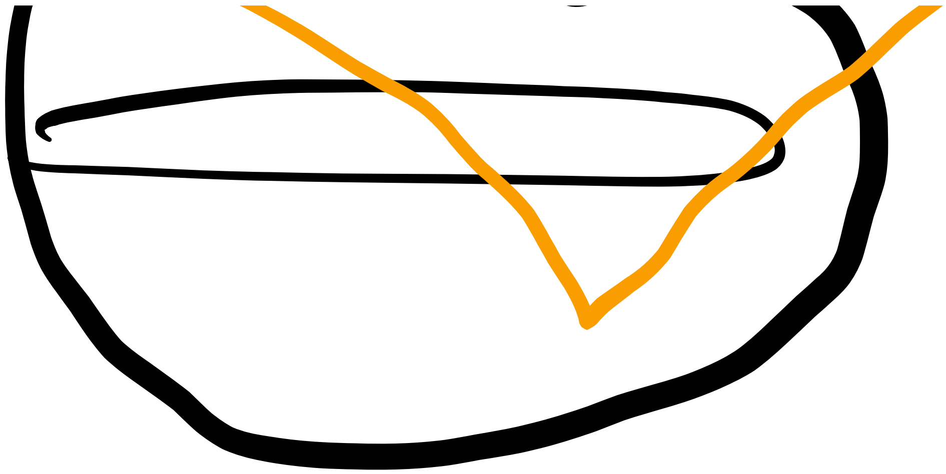
3) Tangent planes

$$x^2+2y^2+z^2=4$$

Find the tangent plane at $P=(1,1,1)$





$$\nabla f = \begin{bmatrix} 2x \\ 4y \\ 2z \end{bmatrix}$$

$$\nabla f(1,1,1) = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

Tangent plane!

$2x + 4y + 2z$

$= d$

plug in $(1,1,1)$

$= 8$

4 Directional derivative

$$\boxed{D_{\vec{v}} f = \nabla f \cdot \vec{v}}$$

directional derivative

[illegible]

for the

direction $\vec{v}$

= unit vector.

<u>Interpretation:</u>

Slope in the

direction $\vec{v}$

Example:

Example:

$$f = 4500 - x^2 - y^2$$

$$\vec{v} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$(x_0, y_0) = (10, 20)$$

$$D_v f(10, 20)$$

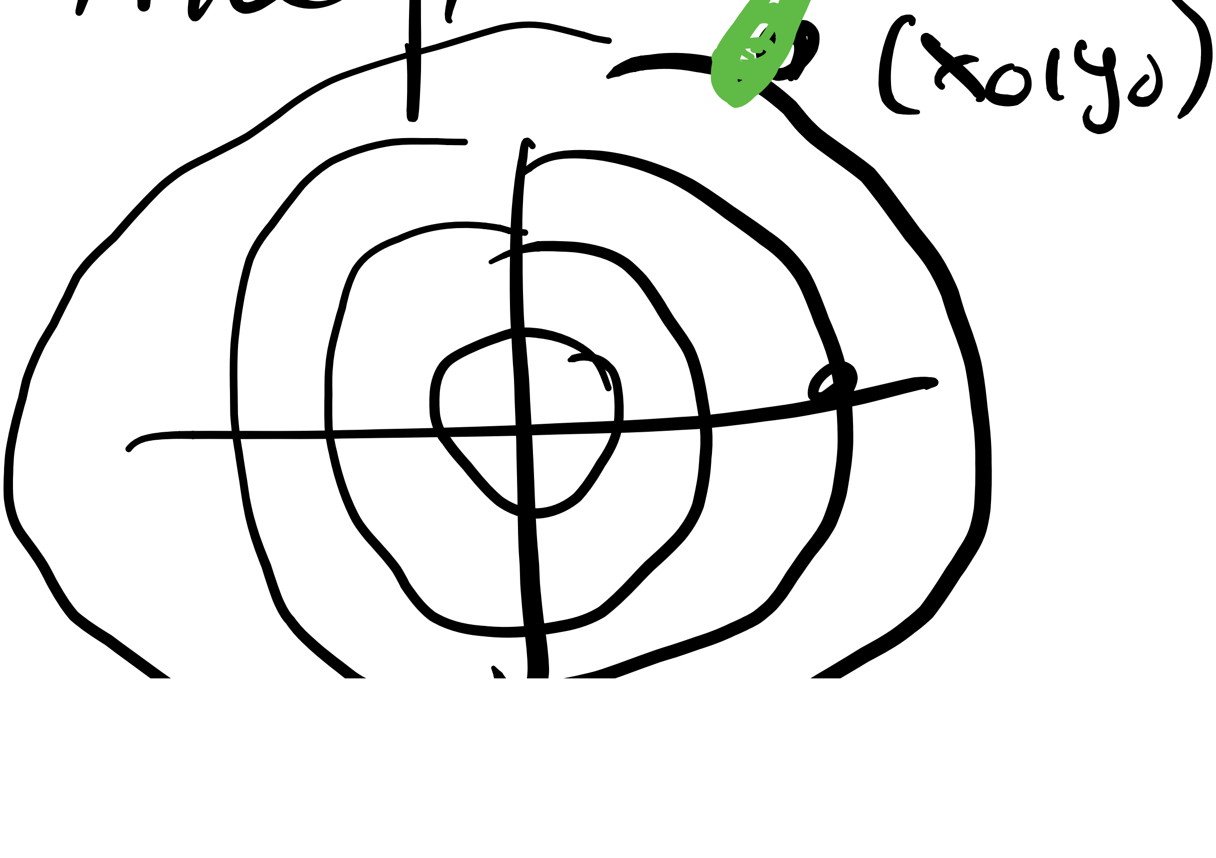
$$= \begin{bmatrix} -2 \cdot 10 \\ \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ \end{bmatrix}$$

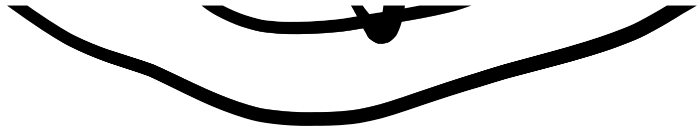
$[-2 \cdot 20] \ [4/5]$

$= -12 - 32$

$= \boxed{-44}$

Interpretation:





we are going down the mountain!

$$D_{\begin{bmatrix}1\\0\end{bmatrix}} f(x,y)$$

$$\nabla f \cdot \begin{bmatrix}1\\0\end{bmatrix}$$

$$= f_x$$

Slope when moving in the x direction

"East"

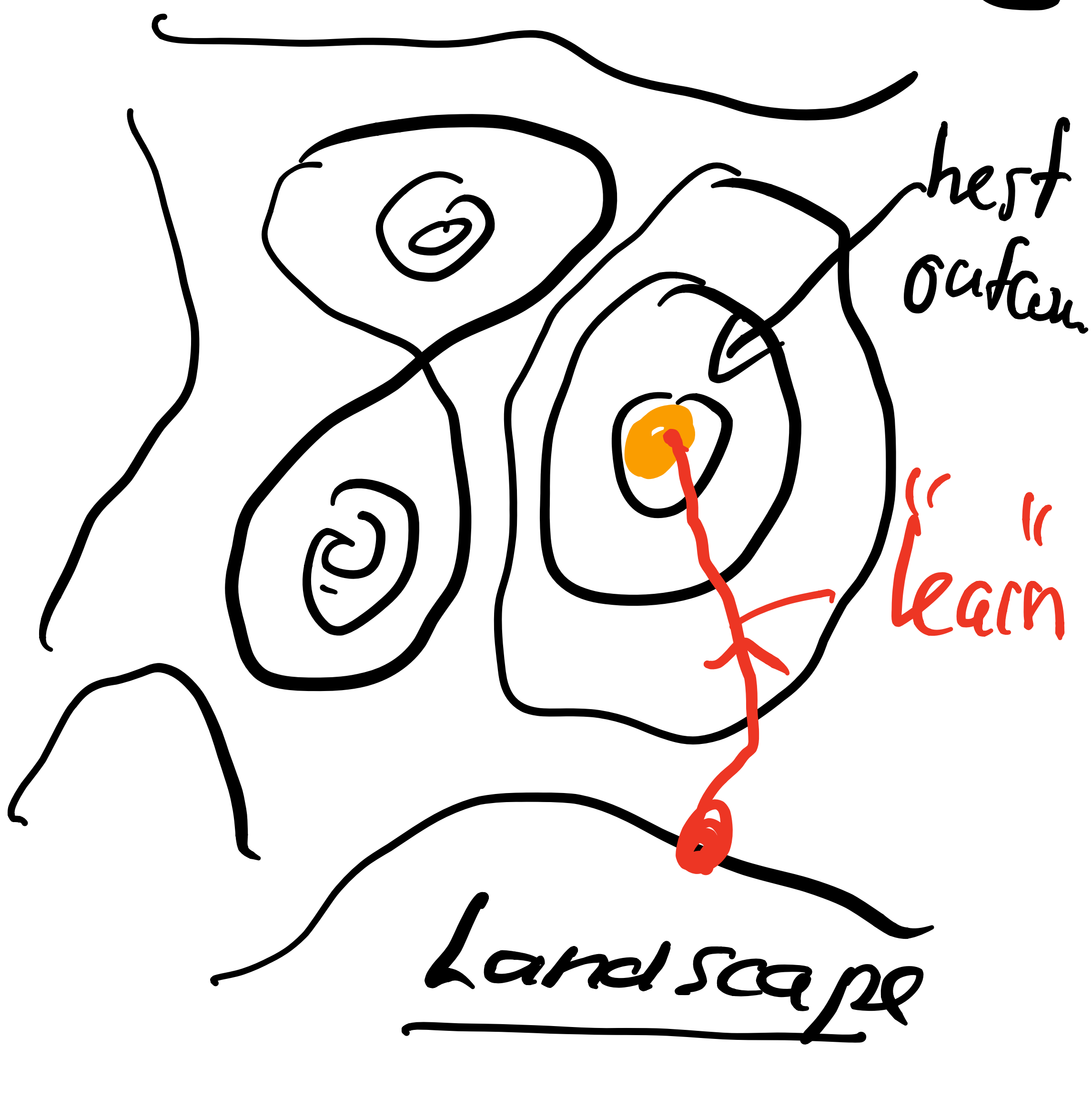
$$D_{\begin{bmatrix}0\\1\end{bmatrix}} f(x,y) = f_y(x,y)$$

5) Steepest ascent

In which direction do we have to move to increase f most?

used in

machine learning



$$D_{\frac{\nabla f}{|\nabla f|}} f =$$

(directional derivative in the direction of the gradient)

$$\nabla f \qquad \frac{\nabla f}{|\nabla f|}$$

unit vector

$$\frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

$|\nabla f|$ is the maximal

Steepness.

And if we go in the direction of the gradient, the slope is $|\nabla f|$

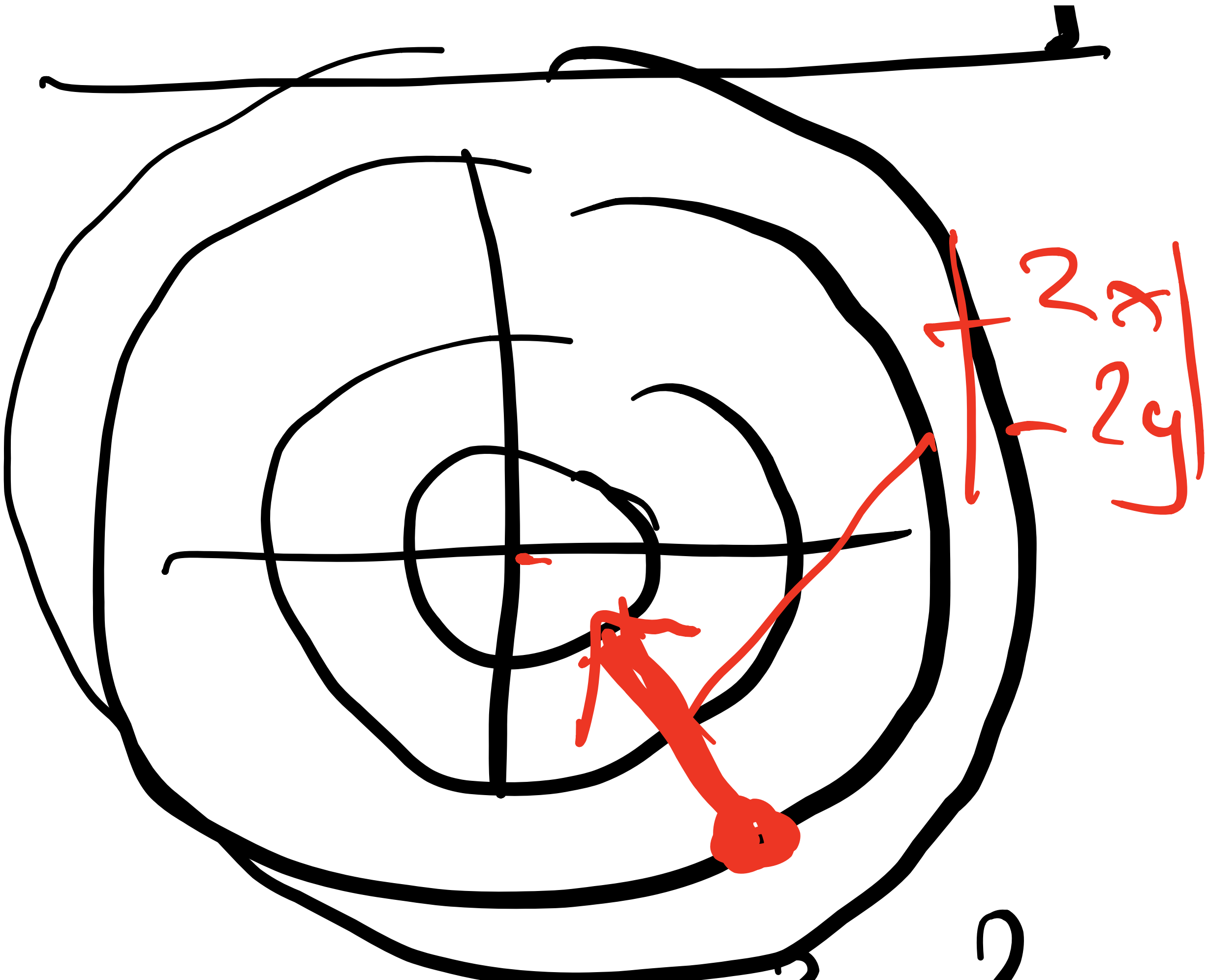
$$|D_v f|$$

$$= \nabla f \cdot \vec{v}$$

$$= |\nabla f|\,|v|\cos\alpha$$

$$= |\nabla f| \cdot \cos\alpha$$

$$\leq |\nabla f|$$



$2450 - x^2 - y^2$

Summary of unit 1?

of a ...

- $\nabla f \perp \{f = c\}$
- $|\nabla f|$ is the **maximal steepness**
- ∇f **points "upwards"**

"Angels dance upwards"

If $|\nabla f| = \overline{0}$

then we

are on the

top

$\longrightarrow$ next

week

Extrema !