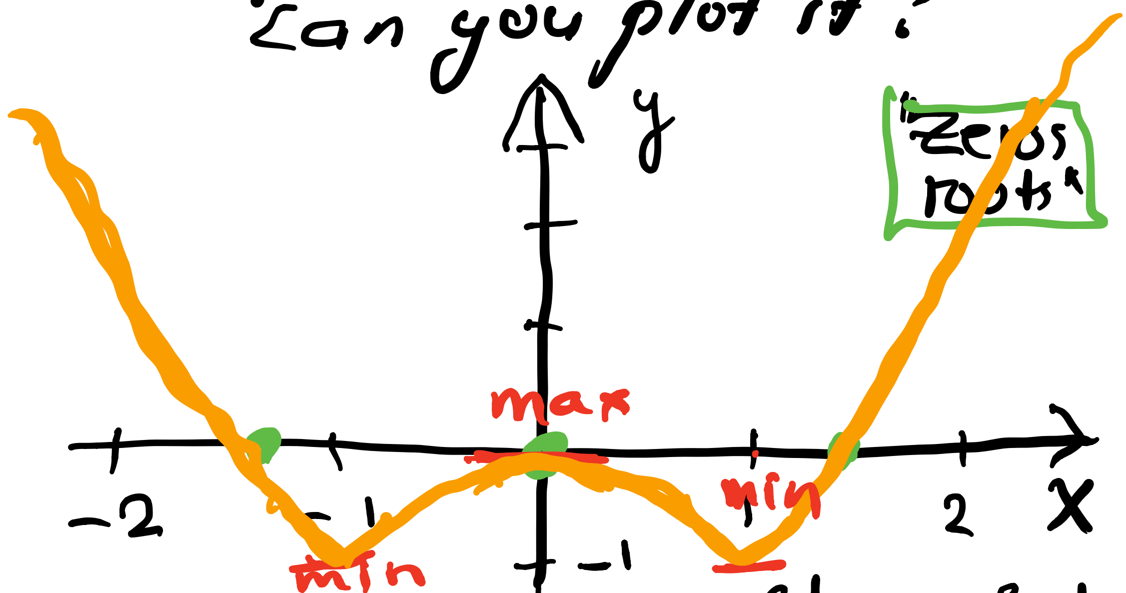
Unit 13

(1) 1D review

$$f(x) = x^4 - 2x^2$$

Can you plot it?



$f''(x) = 12x^2 - 4$

$f'' = -4$ at 0

$= 8$ at ± 1

$f' = 4x^3 - 4x$

$= 4x(x^2-1)$

$= 0$

critical point

"W" "Goldstone boson"

"Symmetry breaking"

② **Critical points**

Fermat: if $\nabla f \neq [0,0]$, then we can increase f by going in the direction of ∇f

$$D_{\frac{\nabla f}{|\nabla f|}} f(x) = \nabla f \cdot \frac{\nabla f}{|\nabla f|}$$

$$= \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f| > 0$$

So, for a local max or min, we need $\nabla f = [0,0]$

Def: $\boxed{(x,y) \text{ is a critical point of } f, \text{ if } \nabla f(x,y) = [0,0]}$

Examples: 3 basic

$f(x,y)=x^2+y^2$ | $f(x,y)=y^2-x^2$ | $f(x,y)=-x^2-y^2$

$\nabla f=[2x,2y]$

min

saddle

$\nabla f=[2x,2y]$

max

robust situation, "Morse"

local minimum = min
local maximum = max

③ **Second derivative test**

Define

$$D=f_{xx}f_{yy}-f_{xy}^2$$

Discriminant.

$$= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$= \det$ Hessian

Theorem 2. deriv test

(x_0, y_0) is a crit. point.

a) if $D > 0$, $f_{xx} < 0 \Rightarrow$ max

b) if $D > 0$, $f_{xx} > 0 \Rightarrow$ min

c) if $D < 0 \Longrightarrow$ Saddle

mini saddl max

→ Discussion

f', $\nabla f = [f_x, f_y]$

f'', $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$
matrix

→ $D = f_{xx} f_{yy} - f_{xy}^2$

If D is not zero for every crit point, the f is called Morse

D Curvature

→ mass

Sitter
universe

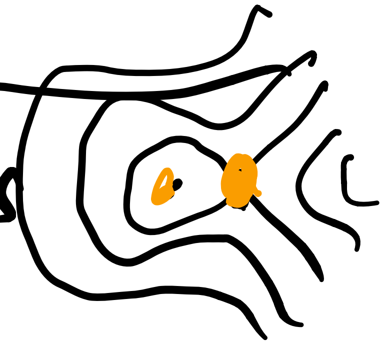
Anti
de Sitter
AdS

$D > 0,\ f_{xx} < 0$

$f_{xx} f_{yy} - f_{xy}^2 > 0$

Can not happen

(4) Examples



a) $f(x,y) = x^4 - 4x + y^3 - 3y$

$$\nabla f = [\overset{\partial x}{4x^3 - 4},\ \overset{\partial y}{3y^2 - 3}]$$

$$= [0, 0]$$

$$\Rightarrow x^3 = 1,\ y^2 = 1$$

$$(1, 1),\ (1, -1)$$

Points	D	f_{xx}	Nature	f
(1,1)	72	12	Min	5
(1,-1)	-72		Saddle	

$$f_{xx} = 12x^2$$
$$f_{yy} = 6y$$
$$f_{xy} = f_{yx} = 0$$
$$D = 12x^2\, 6y = 72x^2y$$

b) A tough example.

$$\boxed{\begin{array}{l} 4xy - x^3y - xy^3 \\ f(x,y) \end{array}}$$

$$f_x = 4y - 3x^2y - y^3$$

$$f_y = 4x - x^3 - 3xy^2$$

$$\begin{array}{ll} ① & y(4 - 3x^2 - y^2) = 0 \\ ② & x(4 - x^2 - 3y^2) = 0 \end{array}$$

Important: factor out if possible

4 cases:

(i) $y = 0$ solves ①

plug into ② gives

plug into 2 gives:

2 $4 - x^2 = 0$

So: $x = \pm 2$

(ii) $x = 0$ solves 2

plug into 1 gives

$4 - y^2 = 0$

2 So: $y = \pm 2$

(iii) $x \neq 0$, $y \neq 0$. we can divide by x, y

4 $$\begin{vmatrix} 4 - 3x^2 - y^2 = 0 \\ 4 - x^2 - 3y^2 = 0 \end{vmatrix}$$

System of equations for x^2, y^2 gives $x^2 = 1$, $x = \pm 1$; $y^2 = 1$, $y = \pm 1$

(iv) $x=0, y=0$ solves both equations

Point	D	f_{xx}	Nature
(0,2)	<0	~~×~~	Saad
(0,-2)	<0	~~×~~	Saadl
(2,0)	<0	~~×~~	Saadle
(-2,0)	<0	~~×~~	Saadle
(1,1)	32	<0	max
(1,-1)	32	>0	min
(-1,1)	32	>0	min
(-1,-1)	32	<0	max
(0,0)	<0	~~×~~	Saadle

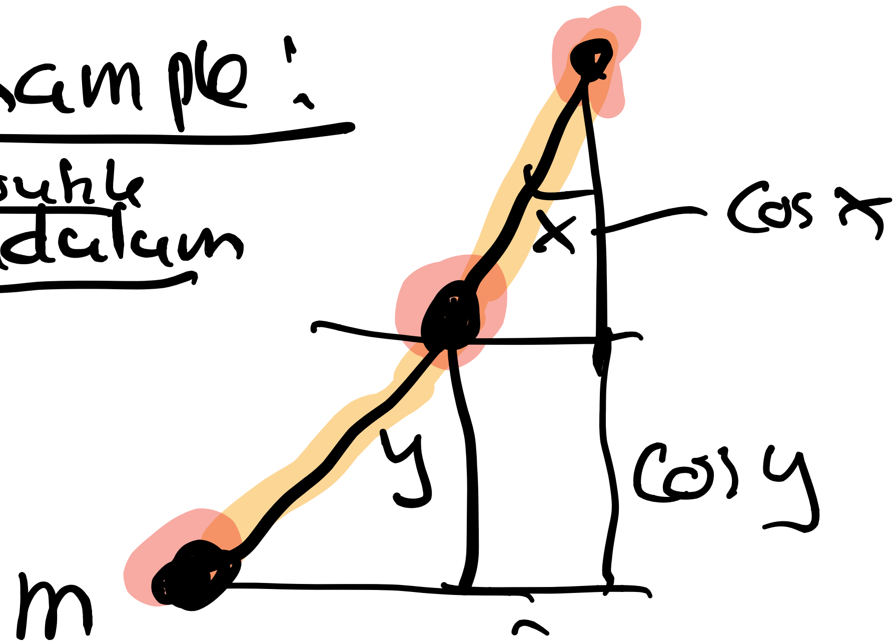
$f_{xx} = -6xy$, $f_{xy} = 4-3x^2-3y^2$

$f_{yy} = -6xy$,

$$D = 36x^2y^2 - (4-3x^2-3y^2)^2$$

Example:

double pendulum



$$f(x,y) = -\cos x - \cos y$$

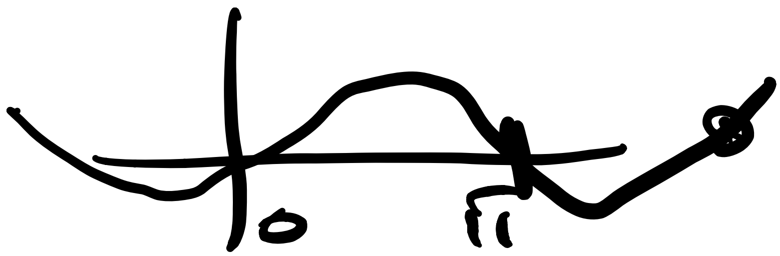
height

Classify the crit. points

$$\nabla f(x,y) = [\sin x, \sin y]$$

Point	D	f_{xx}	Nature
(0,0)	1	1	min
(0,π)	−1		Saddle
(π,0)	−1		Saddle
(π,π)	1	−1	max

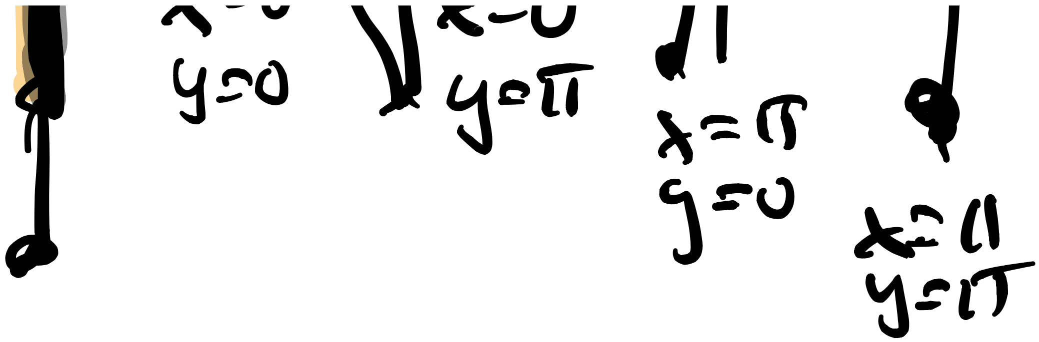
$\sin x = 0$ at $0, \pi$



$f_{xx} = \cos x$ $\quad f_{xy} = 0$

$f_{yy} = \cos y$

$D = \cos(x)\cos(y)$



$\boxed{D=0}$ $f_{xx}f_{yy}=f_{xy}^2$

$f=3$

<u>every point is critical</u>

e^{xy} f_{xx}

$f_x=[ye^{xy}, xe^{xy}]$

$2 - x^2 - y^2$

$D = 4 > 0$

$2 - (x+y)^2$

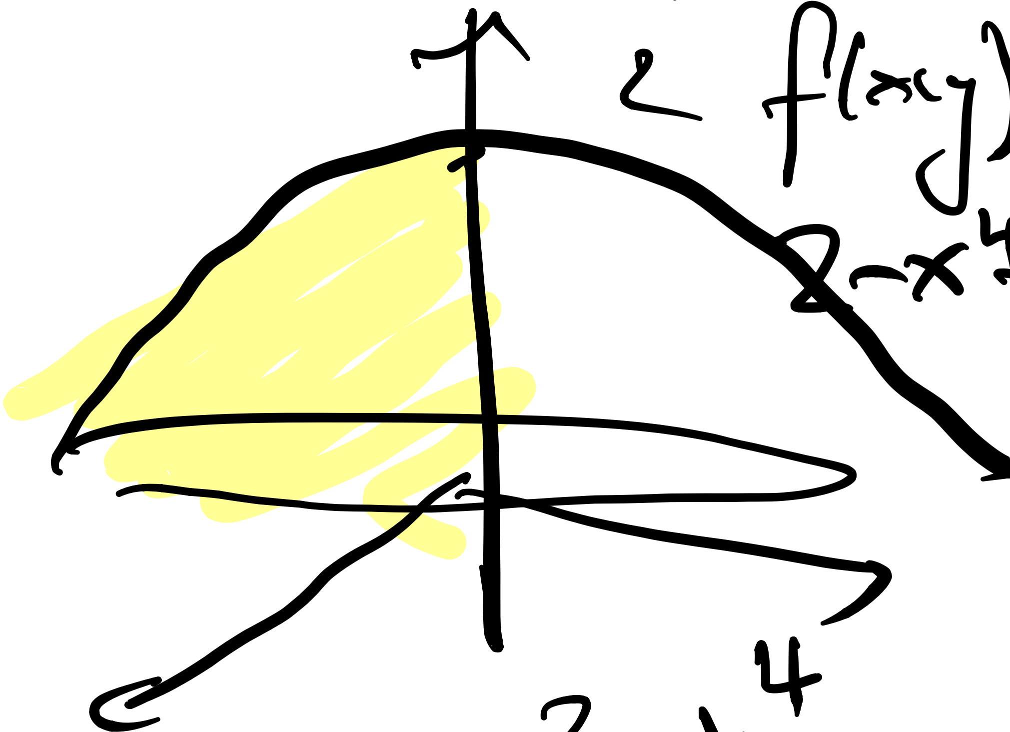
$f_{xx} = -2$ $D > 0$

$f_{yy} = -2$

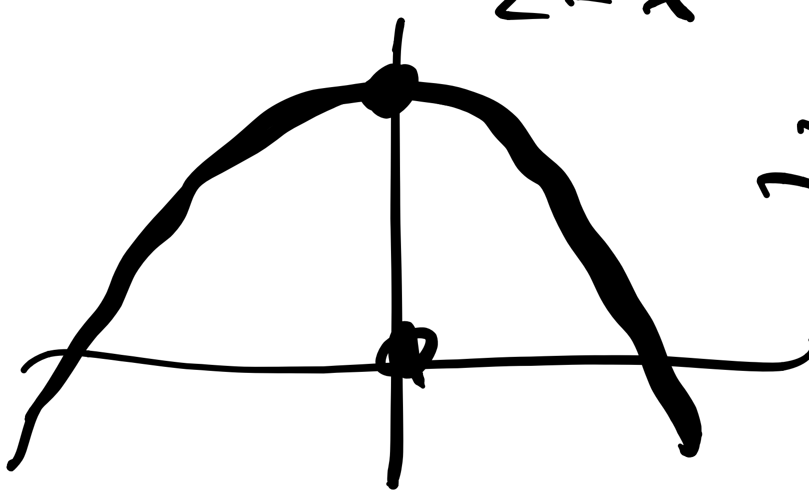
$f\,x\,y =$

$z\; f(x,y)$

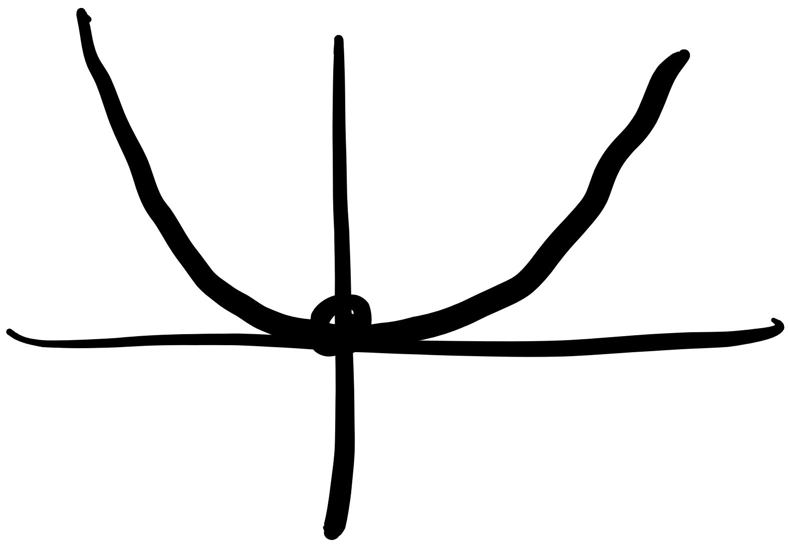
$2-x^4 y^4$



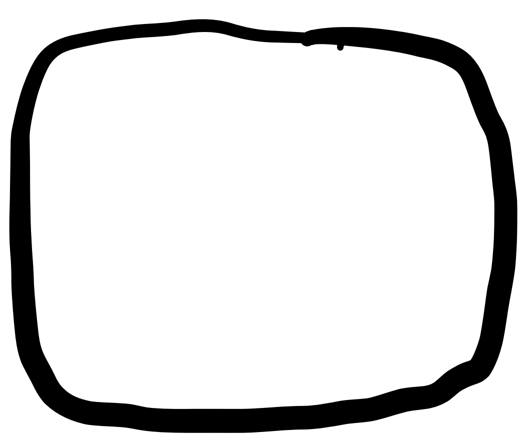
$2-x^4$



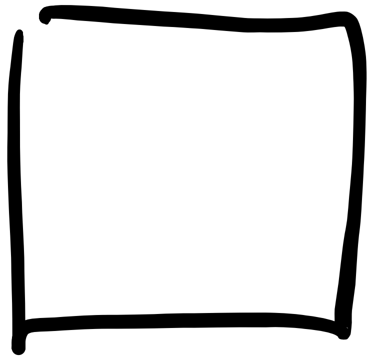
$f''(0)$
$=0$



$$x^4 + y^4 = 1$$



$$x^{1000} + y^{1000} = 1$$



$x^{1000} + y^{100}$