Unit 14 Lagrange

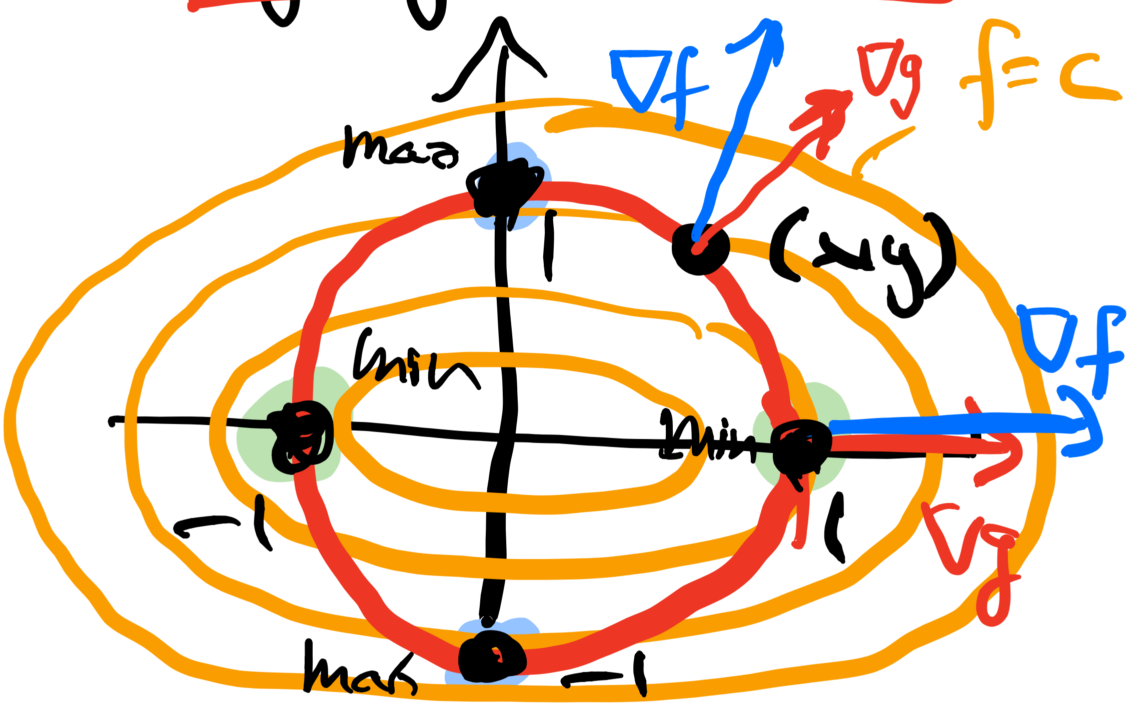
① Lagrange eqns

Maximize or minimize

$f(x,y) = x^2+y^2 \cdot 4$

under the constraint

$$\boxed{g(x,y) = x^2+y^2-1} = 0$$



$\nabla g = [2x, 2y]$

$\nabla f = [2x, 8y]$

In order that f is maximal or minimal, we need ∇f to be parallel to ∇g. (Lagrange)

$$\boxed{\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= c \end{aligned}}$$

<u>Lagrang equations</u>

λ <u>Lagrange multiplier</u>

"Lambda".

These are three equations for three unknowns

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= c \end{aligned}}$$

<u>In the example</u> :

$f = x^2 + 4y^2$
$g = x^2 + y^2$

$$\left|\begin{array}{ll} ① & 2x = \lambda\, 2x \\ ② & 8y = \lambda\, 2y \\ ③ & x^2 + y^2 = 1 \end{array}\right|$$

Important point :
Eliminate λ first.

$$\frac{①}{②} \qquad \frac{x}{4y} = \frac{x}{y}$$

Cross multiply

$xy = 4xy$

$\boxed{3xy = 0}$

This means : $x = 0$ or $y = 0$

$x = 0$ then ③ : $y = \pm 1$

$y = 0$ then ③ ! $x = \pm 1$

So we have 4 crit. poin

$f = 4$ $f = 1$

$$\boxed{\begin{matrix} (0,1) & (0,-1) \\ (1,0) & (-1,0) \end{matrix}}$$

no second test.

min

What does λ mean.

nothing in gener.

$g(x,y) = 0$

$- \quad g(x,y) = 0$

$g(x,y)$

In economics:

if $g(x,y)=0$

$F(x,y,\lambda)=f(x,y)-\lambda g(x,y)$

Look for crit. points of F (no cons)

$F_x=0$

$F_y=0$

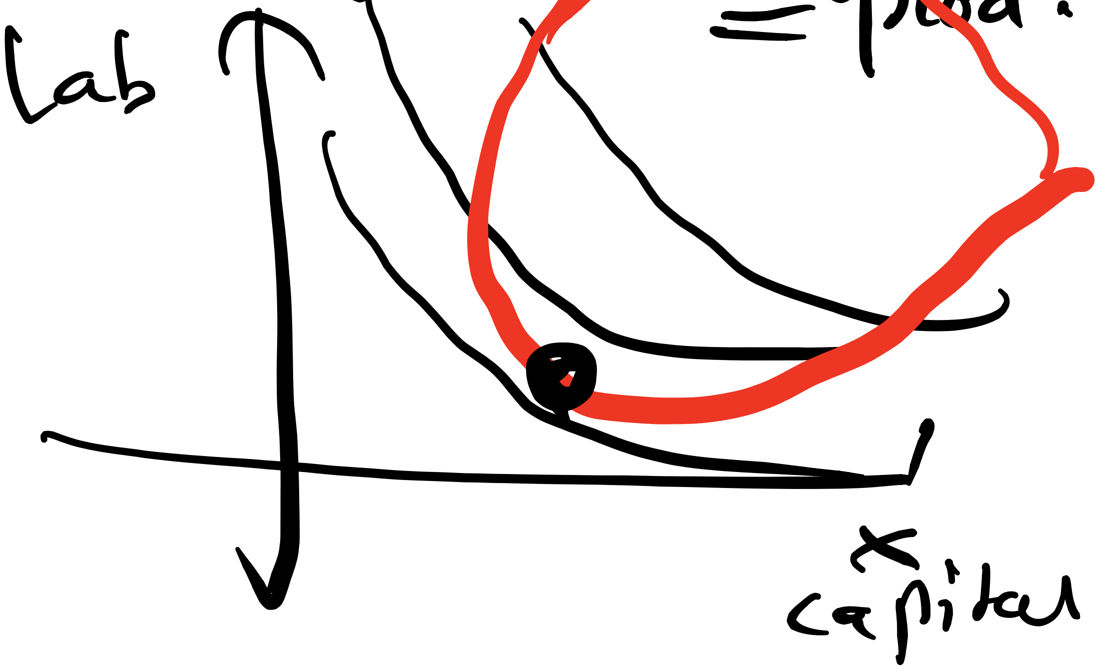
$F_\lambda=0$

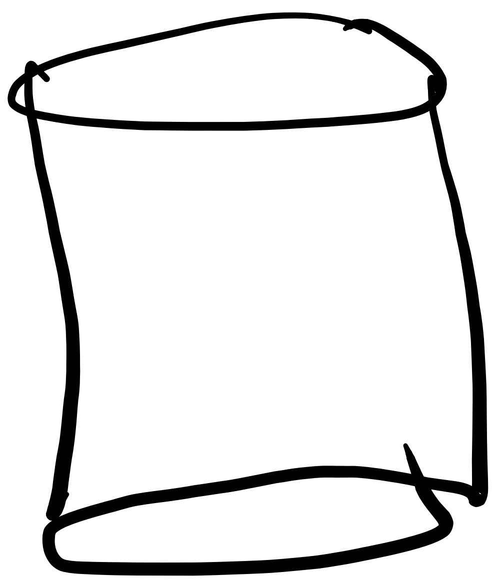
$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= 0 \end{aligned}}$$

Cobb-Douglas

$f(x,y) = x^{\alpha} y^{\beta}$

$= y$ prod.



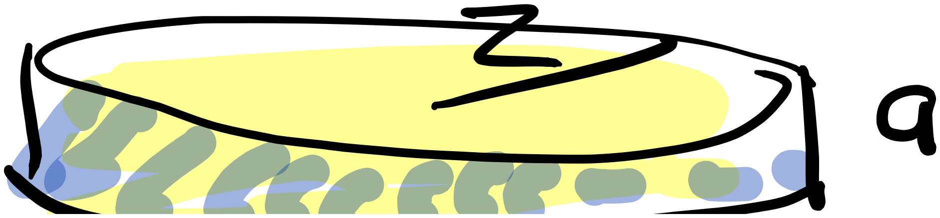


new sect

② Coke problem

$$V = \pi r^2 h = \pi$$

$$A = 2\pi r h + 2\pi r^2$$



Volume = Pizza

Find the geometry
with minimal
Aluminum cost

150 PSI pounds per inch

Lagrange equations

$$f = A = 2\pi r h + 2\pi r^2$$
$$g = V = \pi r^2 h = \Pi$$

constraint

r, h are the
variable.

$$\begin{cases} (1) & 2\pi h + 4\pi r = \lambda\, 2\pi r h \\ (2) & 2\pi r = \lambda \pi r^2 \\ (3) & \pi r^2 h = \pi \end{cases}$$

$$\frac{(1)}{(2)} \quad \frac{2\cancel{\pi} h + 4\cancel{\pi} r}{2\cancel{\pi} r} = \frac{2\cancel{\pi} r h}{\cancel{\pi} r^2}$$

$$\boxed{(2h+4r)\cancel{r^2} = 4\cancel{r^2} h}$$

Is it possible that $r=0$

<u>No!</u> (3) $0=\pi$ ↯

$$2h + 4r = 4h$$

$$4r = 2h$$

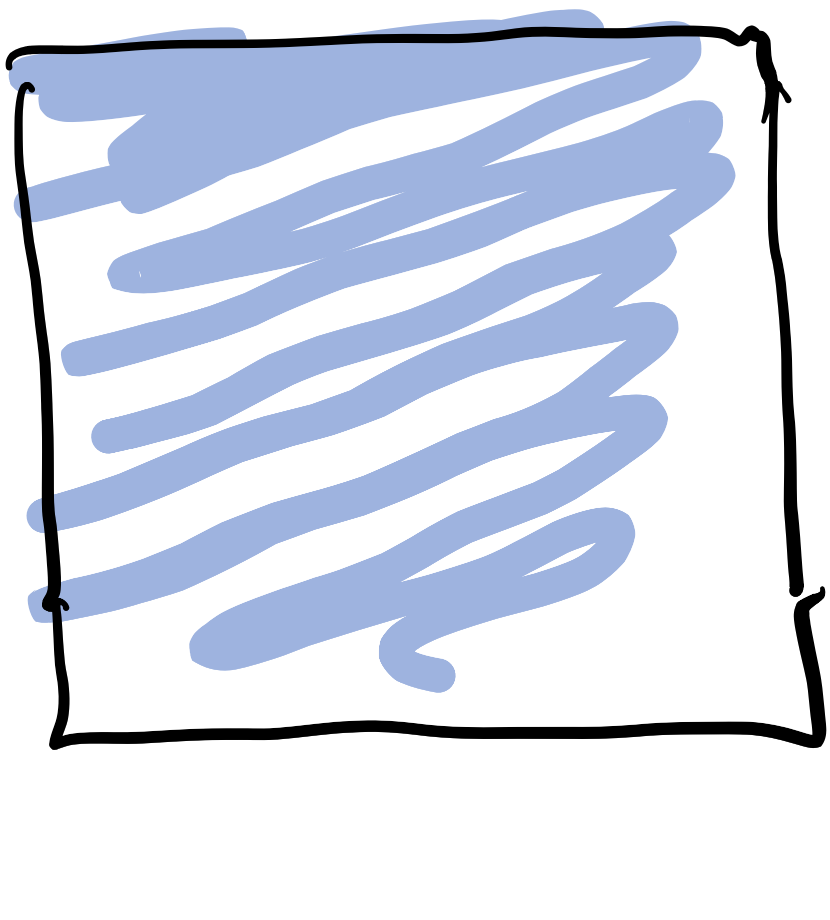
$$\boxed{2r = h}$$

Substit. into (3)

$$\cancel{\pi} r^2 (2r) = \cancel{\pi}$$

$$r^3 = \frac{1}{2}$$

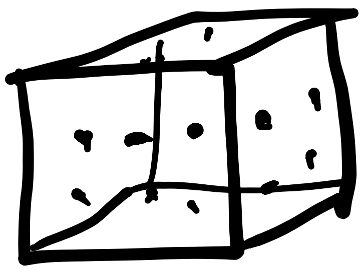
$$\boxed{\begin{aligned} r &= \sqrt[3]{\frac{1}{2}} \\ h &= 2\sqrt[3]{\frac{1}{2}} \end{aligned}}$$



How do we
test min or
max.

Compute the
value at this
point and
compare

3 Entropy



dice

p_k prob. that face k appears

there are 6 variables

$$\boxed{\begin{aligned} S &= -p_1 \log p_1 - p_2 \log p_2 \cdots - p_6 \log p_6 \\ &= -\sum_{k=1} p_k \log p_k \end{aligned}}$$

entropy — measure of disorder

Claud Shannon: Ludwig Bolzman

father of information theory

There is a constraint! What is it.

$g = p_1 + p_2 + \dots + p_6 = 1$

<u>Lagrange problem:</u>

$$\frac{d}{dx} - x \log x = -\log x - 1$$

$$\frac{\partial S}{\partial p_1} = \lambda \cdot \frac{\partial g}{\partial p_1} = \lambda$$

$$\frac{\partial S}{\partial p_2} = \lambda$$

$$\vdots$$

$$\frac{\partial S}{\partial p_6} = \lambda$$

$$
\left|
\begin{array}{l}
-\log p_1 - 1 = \lambda \quad (1) \\
-\log p_2 - 1 = \lambda \quad (2) \\
\quad\vdots \\
-\log p_6 - 1 = \lambda \\
p_1 + p_2 + \dots + p_6 = 1 \quad (7)
\end{array}
\right.
$$

eliminate λ !

$(1) = (2)$ $\quad \log p_1 = \log p_2$

$\qquad\qquad p_1 = p_2$

$(1) = (3)$ $\quad \log p_1 = \log p_3$

$\qquad\qquad p_1 = p_3$

$\vdots$

$(1) = (6)$ $\quad \log p_1 = \log p_6$

$\qquad\qquad p_1 = p_6$

All the p are the same

$p_1 + p_2 + \dots + p_6 = 1$

Therefore all $p_n = \frac{1}{6}$

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fair dice

maximal entropy

More important

term

entropy

$U - TS =$

energy

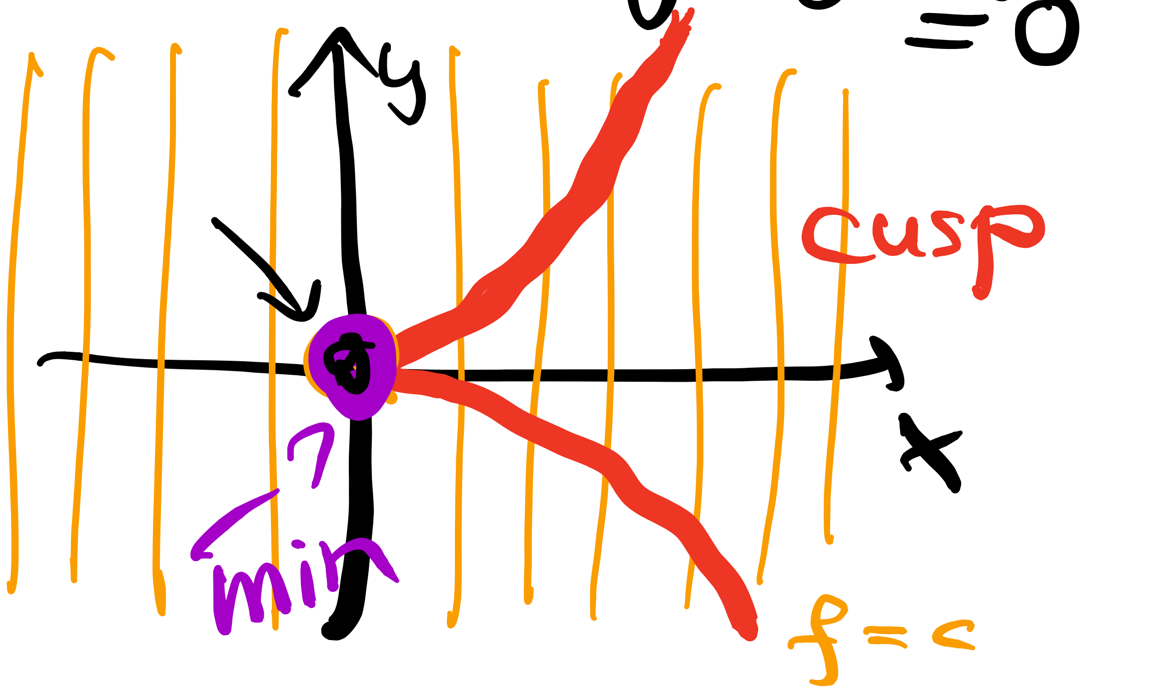
free energy

minimize free energy

(4) A model

Example: $f(x,y) = x$

$g(x,y) = y^2 - x^3 = 0$



There is a nice minimum $x=0$, $y=0$

Solve the Lagrange equations!

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ y^2 - x^3 = 0 \end{cases}$$

$$\begin{cases} 1 = -\lambda 3x^2 & (1) \\ 0 = \lambda 2y & (2) \\ y^2 - x^3 = 0 & (3) \end{cases}$$

$$\frac{②}{①} \quad \frac{0}{1} = \frac{2y}{3x^2}$$

$$0 = 2y \quad \boxed{y = 0}$$

$$③ : \quad \boxed{x = 0}$$

$$① : \quad 1 = 0 \quad ↯$$

The Lagrange equations might miss som points.

$$\nabla g(0,0) = [0,0]$$

The $\vec{0}$ vector is $\parallel$ any other

$$\nabla f \parallel \nabla g$$

$$\Updownarrow$$

$$\nabla f = \lambda \nabla g$$

Lagrange theorem: if $f(x_0, y_0)$ is a max or min under the constraint $g = c$, then

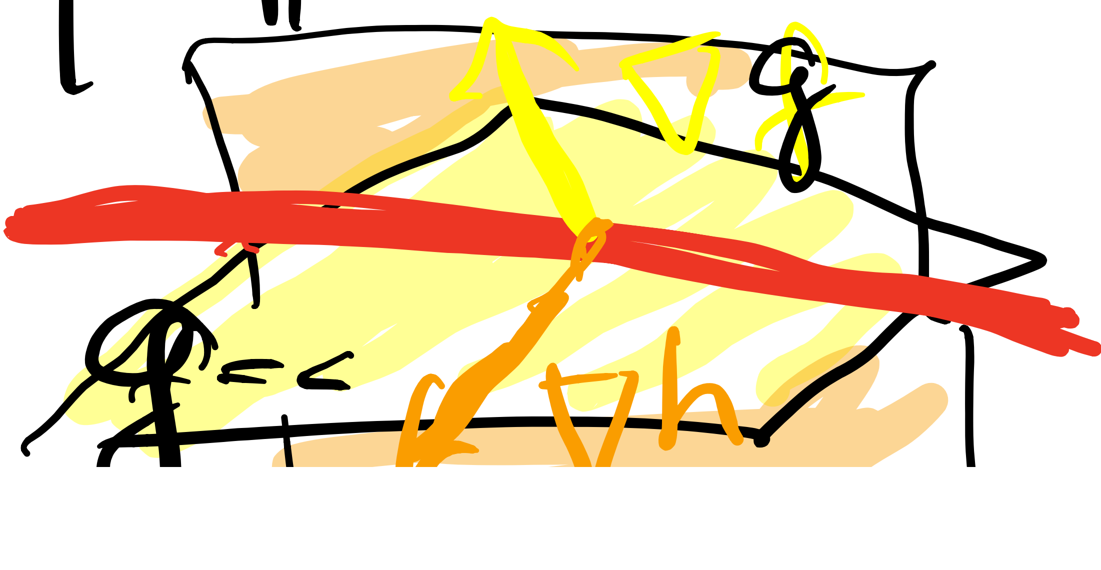
$\boxed{\nabla f = \lambda \nabla g}$ or $\boxed{\nabla \ldots}$

$g = c$ ... 0 or $\boxed{\nabla g = 0}$

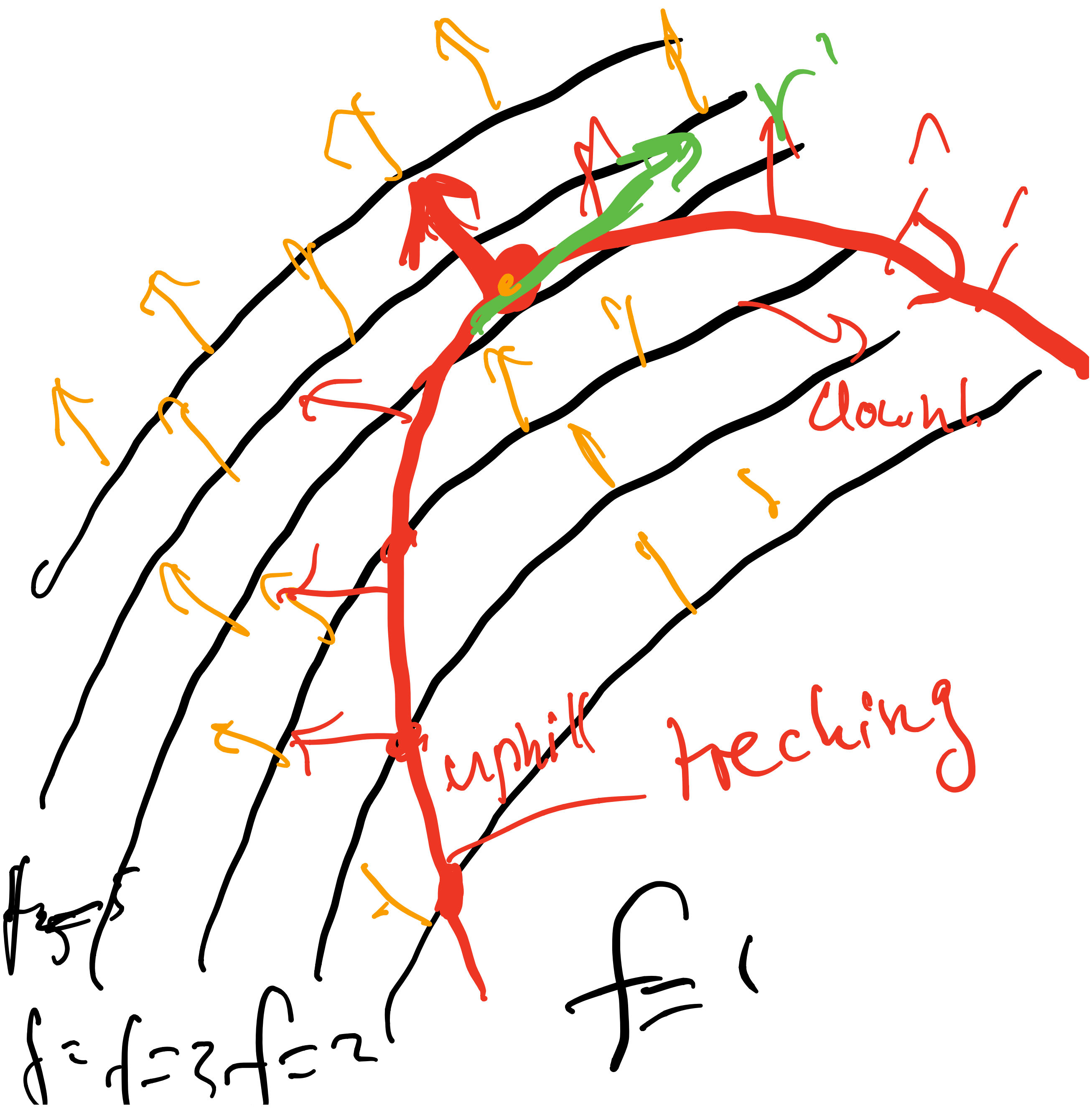
The method also w. more constraints

$$\left[\begin{array}{l} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = c \\ h = d \end{array}\right.$$

μ = "mu"



h=d



$$\frac{d}{dt} f(r(t)) = 0$$

$$\boxed{\nabla f} \cdot \boxed{r'(t)} = 0$$

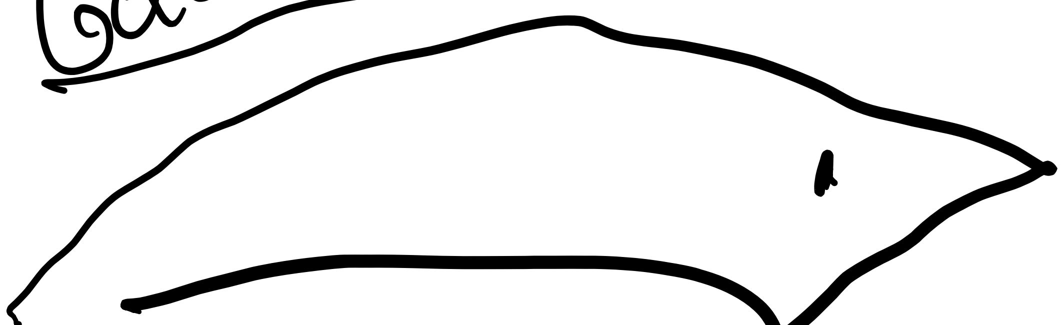
$$\nabla f = \lambda \nabla g$$

λ "Lambda"

μ "Mue"

$$\Pi = LK$$

Gauss curvature



$$z = f(x,y)$$

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{\sqrt{1 + f_x^2 + f_y^2}} =$$

M

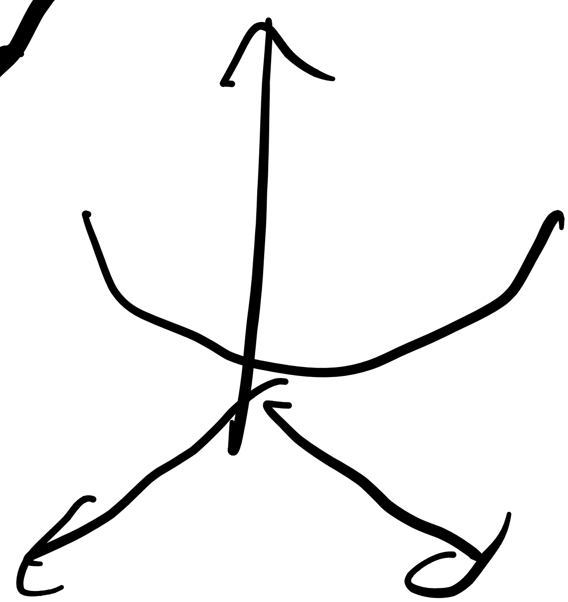
$$\iint_M k \, dA = 0$$

$\iint k \, dA$

$S = 2$

Gauss Bonnel

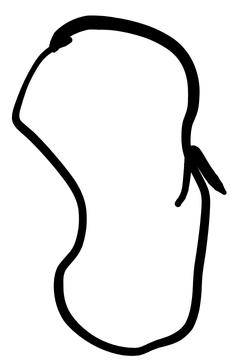
1D



2-Surface

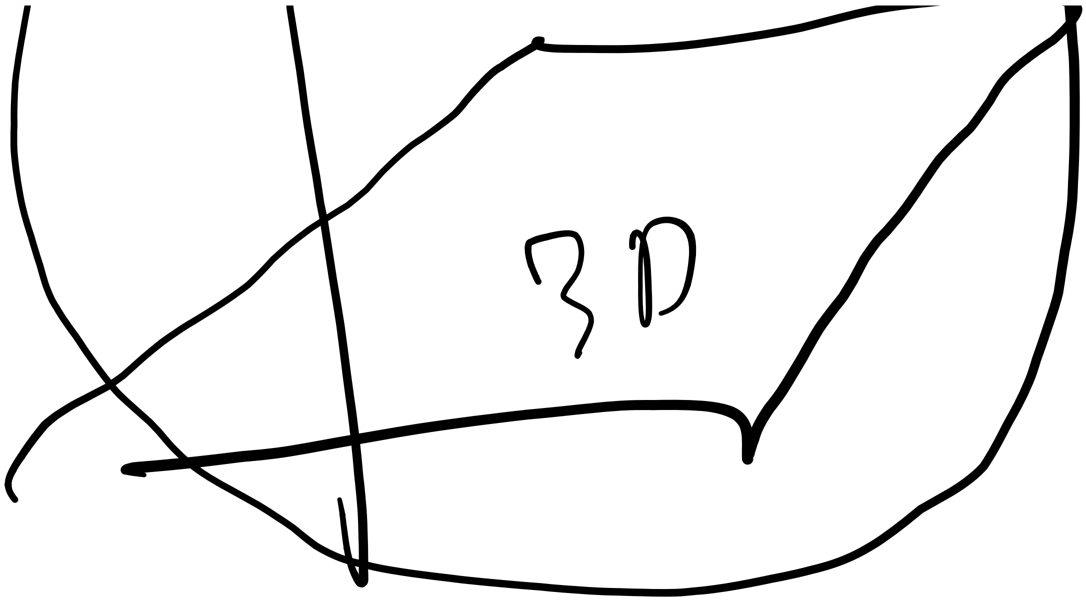
Curve

+ 1D



Branes
Strings


dim



4 dim

∇g is not defin

$$g = |x| - y = 0$$